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AN EXTENSION OF RAMSEY'S THEOREM TO MULTIPARTITE GRAPHS

by

Brian Cook

Under the Direction of Guantao Chen

ABSTRACT

Ramsey Theorem, in the most simple form, states that if we are given a positive integer l, there exists a minimal integer $r(l)$, called the Ramsey number, such any partition of the edges of $K_{r(l)}$ into two sets, i.e. a 2-coloring, yields a copy of K_l contained entirely in one of the partitioned sets, i.e. a monochromatic copy of K_l. We prove an extension of Ramsey's Theorem, in the more general form, by replacing complete graphs by multipartite graphs in both senses, as the partitioned set and as the desired monochromatic graph. More formally, given integers l and k, there exists an integer $p(m)$ such that any 2-coloring of the edges of the complete multipartite graph $K_{p(m);r(k)}$ yields a monochromatic copy of $K_{m;k}$.

The tools that are used to prove this result are the Szemerédi Regularity Lemma and the Blow Up Lemma. A full proof of the Regularity Lemma is given. The Blow-Up Lemma is merely stated, but other graph embedding results are given. It is also shown that certain embedding conditions on classes of graphs, namely (f, δ)-embeddability, provides a method to bound the order of the multipartite Ramsey numbers on the graphs. This provides a method to prove that a large class of graphs, including trees, graphs of bounded degree, and planar graphs, has a linear bound, in terms of the number of vertices, on the multipartite Ramsey number.

INDEX WORDS: Ramsey numbers, Multipartite Ramsey numbers, Regularity Lemma, Blow-Up Lemma, p-arrangeable, d-degenerate.

AN EXTENSION OF RAMSEY'S THEOREM TO MULTIPARTITE GRAPHS

by

Brian Cook

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2007

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To Tigger and Viktoriya

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Chapter 1

Introduction and Notation

1.1 Notation

All graphs are simple, the number of verticies in a graph, say $G = (V, E)$, is denoted by $|G|$, and the number of edges is denoted by $||G||$. When referring to the vertex set or the edge set of some graph, we write $V(G)$ or $E(G)$ respectively. The degree of a vertex, say v, is denoted by $\deg(v)$. The maximal degree of a vertex in a graph G is denoted by $\Delta(G)$. The average degree over all vertices is given by $d(G)$.

K_k denotes the complete graph on k vertices. The graph $K_{n;k}$ is the complete k-partite graph with n vertices in each partite class. An embedding of a graph G into H is a map $f : V(G) \to V(H)$ such that if $v_1v_2 \in E(G)$, then $f(v_1)f(v_2) \in E(H)$. If G and H are graphs and there exists an embedding of H into G, we say that G is embeddable into H, and that G is a subgraph of H, which is denoted by $G \subseteq H$. The chromatic number of a graph G is denoted by $\chi(G)$, and is given by the minimal integer k such that $G \subseteq K_{|G|;k}$. For a particular graph $G = (V, E)$, we define an edge coloring with c colors to be a map

$$\chi : E \to \{1, 2, ..., c\},$$

which is referred to as a c-coloring on G. There is some abuse on the use of χ, but no confusion should come from this. We introduce the vertex set $V_k(n) = \{V_1, V_2, .., V_k\}$ as the k-partite vertex set with $|V_i| = n$ vertices in each class. A graph on $V_k(n)$, say G, is given by $G = (V_k(n), E)$, where this means $G \subseteq K_{n;k}$. Given such a G, it is supposed that $|G| = nk$.

The rest of the notation that we use is mostly standard and should be understood by all. For example, $\mathbb{R}$ is the set of real numbers, $(0, 1)$ is the open unit interval, $\lfloor x \rfloor$ is the floor function, etc.

1.2 Background

This work is focused mainly on an extension of Ramsey's theorem, which is the centerpiece of a branch of combinatorics called Ramsey theory. This theorem and the proof are particularly nice, so we give them together.

(1.2.1) Theorem. *Given natural numbers $k_1, k_2, ..., k_c$, $k_i \geq 2$, there exists a minimal natural number $r(k_1, k_2, ..., k_c)$ such that any c-coloring of $K_{r(k_1,k_2,...,k_c)}$ contains, for at least one i, a monochromatic copy of K_{k_i} in color i.*

Proof. The proof when $c = 2$ is carried out by induction, and then induction is carried out on c.

We see the obvious fact that $r(3, 2) = r(2, 3) = 3$, or more generally $r(2, k) = r(k, 2) = k$ for any $k \geq 2$. Now assume that the theorem is true for all $k_1 \geq 2$ and $k_2 \geq 2$ with $k_1 + k_2 \leq l$, where $l \geq 5$. Consider $k_1 > 2$ and $k_2 > 2$ with $k_1 + k_2 = l + 1$, and let $N = r(k_1 - 1, k_2) + r(k_1, k_2 - 1)$.

Let $\chi : E(K_N) \to \{1,2\}$ be any 2-coloring. Select any vertex in K_N, say v. Define

$$B_1 = \{w : \chi(v,w) = 1\},$$

and B_2 similarly. As $|B_1| + |B_2| = N - 1 = r(k_1 - 1, k_2) + r(k_1, k_2 - 1) - 1$, we must have either $|B_1| \geq r(k_1 - 1, k_2)$ or $|B_2| \geq r(k_1, k_2 - 1)$. Without loss of generality, assume that the former inequality holds. The subgraph of K_N induced by B_1, say H, is two colored by the restriction of χ to $E(G)$. Because G is a complete graph and $|G| \geq r(k_1 - 1, k_2)$, the induction hypothesis guarantees that we have either a monochromatic K_{k_1-1} in color one, or a monochromatic K_{k_2} in color two. If the latter holds, then we are done. Otherwise, in B_1 we have a monochromatic K_{k_1-1} in color one, so that the restricted coloring on the subgraph induced by $B_1 \bigcup \{v\}$ contains a monochromatic K_{k_1} in color one.

Thus we have that the theorem is true for all $k_1, k_2 > 1$ with $k_1 + k_2 = l + 1$. Induction then guarantees that the theorem is true when $c = 2$.

We now proceed with the induction on c. Assume that the theorem is true for $2 \leq c \leq C$, and consider the case $c = C + 1$. Let $k_1, k_2, ..., k_{C+1} \geq 2$ be given, and let $N = r(k_1, r(k_2, ..., k_{C+1}))$. Given any $\chi : E(K_N) \to \{1, 2, ..., C + 1\}$, define $\chi^* : E(K_N) \to \{1,2\}$ by $\chi^*(e) = 1$ if $\chi(e) = 1$, and $\chi^*(e) = 2$ otherwise. Because χ^* is a 2-coloring, we can guarantee that under χ^* we have a monochromatic K_{k_1} in color one, or a monochromatic $K_{r(k_2,...,k_{C+1})}$ in color two. If the former holds, then we are done. Otherwise, define $\chi^{**} : E(K_{r(k_2,...,k_{C+1})}) \to \{1, 2, ..., C\}$ on this $K_{r(k_2,...,k_{C+1})}$, monochromatic under χ^*, by $\chi^{**}(e) = i - 1$ if $\chi(e) = i$. This is well defined. By induction, we have under χ^{**} some monochromatic K_{k_j} in color $j - 1$. This clearly gives a monochromatic K_j in color j under χ. Thus, the case $c = C + 1$ is true, and induction then guarantees that the theorem is true for $c \geq 2$. $\square$

This theorem can be stated in a slightly more general form, which is in fact only a simple corollary to the above.

(1.2.2) Corollary. *Given any graphs $G_1, ..., G_c$, there exists a minimal integer $r(G_1, ..., G_c)$ such that any c coloring of $K_{r(G_1,...,G_c)}$ contains, for at least one i, a monochromatic copy of G_i in color i.*

This minimal integer $r_c(G_1, ..., G_c)$ is called the Ramsey number. When $c = 2$, this is generally given simply by $r(G_1, G_2)$, and the diagonal case, i.e. when all the G_i are the same, is given simply by $r_c(G)$.

This method of proof of this theorem, and in fact all known proofs, give no hint at what this actual number is in general and only a handful of exact values are known. However, this method of proof does give an upper bound to this number, and such bounds become the main focus of study. This is precisely the case with all the loosely knit theorems in Ramsey theory.

In specific reference to Ramsey's theorem, there are numerous conjectures on Ramsey number bounds in explicit cases. There is one in specific that has received some attention that we focus on. The following is due to Burr and Erdős [3]:

(1.2.3) Conjecture. Let $d > 0$. There exists a constant $c = c(d)$ such that any graph G with $d(H) \leq d$ for all $H \subseteq G$ has $r(G) \leq c|G|$.

This conjecture is still open, although there some progress has been made. Chavatal, etc., have proven that a weaker conclusion is true [4].

(1.2.4) Theorem. *Let $d > 0$. There exists a constant $c = c(d)$ such that any graph G with $\Delta(G) \leq d$ has $r(G) \leq c|G|$.*

This result has been improved by Chen and Schelp [5]. We need a definition.

(1.2.5) Definition. Let G be a graph of size n. If there exists an ordering of the vertices $v_1, v_2, ..., v_n$ such that, with $N_{L_i}(v_j) = \{v_k : v_k v_j \in E(G), k \leq i\}$ and $N_R(v_i) = \{v_j : v_i v_j \in E(G), j > i\}$, we have

$$|\bigcup_{v_\alpha \in N_R(v_i)} N_{L_i}(v_\alpha)| \leq p$$

for all $1 \leq i \leq n$, then G is *p-arrangeable.*

Chen and Schelp provide the following result for Ramsey numbers of p-arrangeable graphs:

(1.2.6) Theorem. *Let $p > 0$. There exists a constant $c = c(p)$ such that any graph G that is p-arrangeable has $r(G) \leq c|G|$.*

Theorem **??** is much an improvement to theorem **??**. Given any graph that has all vertices with $\deg(v) \leq d$, then this graph is at worst $(d(d-1)+1)$- arrangeable. Any tree is 1-arrangeable. Chen and Schelp prove that all planar graphs are at worst 761-arrangeable, which has subsequently been improved to 10-arrangeable by Kierstead and Trotter [7]. The set of outer planar graphs is 3-arrangeable, which is also shown by Chen and Schelp.

In a somewhat different direction, there are the subdivisions of graphs. A graph G is a subdivision if there exists a graph H such that G is obtained by replacing the edges of H by paths of length 2. Subdivisions are not contained in the class of p-arrangeable for any fixed p. However, Alon proves a linear bound for class of graphs that contains all subdivisions [1].

(1.2.7) Theorem. *Let G be a graph such that any two vertices of degree at least equal to 3 are not adjacent, then $r(G) \leq 12|G|$.*

These results are still a far cry from the conjecture of Burr and Erdős, but by themselves are still quite interesting.

1.3 Overview

In the next chapter, we discuss one of the more useful tools in all of combinatorics, the Szemerédi Regularity Lemma. A full proof is given, which requires some preliminary work. The first section is dedicated to providing the required definitions and proving some useful lemmas, which include the standard Cauchy-Schwarz inequality and also an altered version with an error term. The second section of this chapter provides the rather lengthy proof of the Regularity Lemma.

Chapter 3 is focused on graph embeddings, specifically embedding graphs into (ϵ, δ)-super-regular graphs (defined in Chapter 2). The main result of this type, about graphs of bounded degree, is formally known as the Blow Up Lemma. The Lemma is stated in the first section, but not proven. The concept of (f, δ)-embeddability is also introduced in this section. The second section of this chapter proves that the complete graphs fit into this concept, while the following section proves that the p-arrangeable graphs are (f, δ)-embeddable with f linear in $|G|$. The final section is a heuristic discussion on the Burr-Erdős conjecture.

Chapter 4 is devoted to the extension of Ramsey's Theorem and the Ramsey-type numbers implied by it, dubbed the multipartite Ramsey numbers. The first section provides the statement of the extension along with the proof. The next section provides bounds on the multipartite Ramsey numbers in terms of (f, δ)-embeddability. More specific bounds are then proven in the next section via alternate proofs of the extension of Ramsey's Theorem. The final section provides lower bounds for these numbers for complete multipartite graphs.

Chapter 2

The Regularity Lemma

2.1 Preliminary Ideas

For two disjoint vertex sets A and B of a graph G, we denote the number of edges between A and B by $e(A, B)$, and the density of A and B is defined by $d(A, B) = \frac{e(A,B)}{|A|\cdot|B|}$, and is necessarily no larger than 1. For such disjoint vertex sets A and B of some graph, we have the following definitions:

(2.1.1) Definition. Let $\epsilon > 0$ be given. A and B are *ϵ-regular* if for any $X \subset A$ and $Y \subset B$ with $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have that

$$|d(A, B) - d(X, Y)| < \epsilon.$$

(2.1.2) Definition. Let $\epsilon > 0$ and $\delta > 0$ be given. A and B are *(ϵ, δ)-super-regular* if for any $X \subset A$ and $Y \subset B$ with $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have that

$$d(X, Y) > \delta,$$

and, given any $a \in A$ and $b \in B$, also that

$$\deg(a) > \delta|B|$$

and

$$\deg(b) > \delta|A|.$$

We also require the following:

(2.1.3) Definition. Let $G = (V, E)$ be a graph and $0 < \epsilon \leq 1$. A partition of V, say $\mathcal{P} = \{X_0, X_1, ..., X_\alpha\}$ is said to be ϵ-uniform if $|X_0| \leq \epsilon|V|$, $|V_i| = |V_j|$ for all $1 \leq i, j \leq \alpha$, and all but at most $\epsilon\alpha^2$ of the pairs (V_i, V_j), $1 \leq i < j \leq \alpha$, are ϵ-regular.

We modify slightly this standard definition; we call an (ϵ, l)-uniform partition to be the same as an ϵ-uniform partition, except that there are allowed to be $\epsilon l\alpha^2$ pairs that are not ϵ-regular.

The first important result that we discuss is Szemerédi's Regularity Lemma. This is actually a variant of the original regularity lemma, but still is similar, and the method of proof, which follows that given in [1], is the same. In short, the original lemma implies that every sufficiently large graph can be almost covered by an ϵ-uniform partition. The version given below implies that l graphs on $V_k(n)$ can simultaneously be almost covered by an (ϵ, l)-uniform partition. The proof requires a few lemmas, the first of which is loosely a continuity result about the density in bipartite pairs.

(2.1.4) Lemma. *If V_1 and V_2 are disjoint vertex sets of a graph G, and $A \subseteq V_1$ with $|A| \geq \alpha|V_1|$, $B \subseteq V_2$ with $|B| \geq \beta|V_2|$, then*

$$|d(X, Y) - d(A, B)| < (2 - \alpha - \beta),$$

and

$$|d^2(X,Y) - d^2(A,B)| < 2(2-\alpha-\beta),$$

where $d^2(X,Y) = (d(X,Y))^2$.

Proof. We have that

$$0 \leq e(V_1,V_2) - e(A,B) \leq |V_1 - A||V_2| + |V_2 - B||V_1| \leq (1-\alpha)|V_1||V_2| + (1-\beta)|V_2||V_1|.$$

Then

$$d(V_1,V_2) - d(A,B) \leq \frac{e(V_1,V_2) - e(A,B)|}{|V_1||V_2|} < (2-\alpha-\beta).$$

This result holds when we consider the graph G^c, the complement of G, which is given on the same vertex set, but only has the edges that are not in G. In G^c, denote the density as d_{G^c}, and denote the density in G by d_G. This satisfies $d_{G^c} = 1 - d_G$, and so

$$d_G(A,B) - d_G(V_1,V_2) = (1 - d_{G^c}(A,B)) - (1 - d_{G^c}(V_1,V_2)) = d_{G^c}(V_1,V_2) - d_{G^c}(A,B).$$

Thus

$$d_G(A,B) - d_G(V_1,V_2) \leq (2-\alpha-\beta),$$

as

$$d_{G^c}(V_1,V_2) - d_{G^c}(A,B) \leq (2-\alpha-\beta).$$

This proves the first result. The last assertion follows easily, as

$$|d^2(X,Y) - d^2(A,B)| = |d(X,Y) - d(A,B)| \cdot |d(X,Y) + d(A,B)| \leq 2(2-\alpha-\beta),$$

where we use the fact that each density is at most one. □

The next lemma we prove is the well known Cauchy-Schwarz inequality.

(2.1.5) Lemma. *Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be two sequences of real numbers. Then*

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2.$$

Proof. Let

$$A = \sum_{i=1}^{n} a_i^2,$$

and

$$B = \sum_{i=1}^{n} a_i b_i.$$

Consider

$$\sum_{i=1}^{n} (\frac{B}{A} a_i - b_i)^2.$$

We have that

$$0 \leq \sum_{i=1}^{n} (\frac{B}{A} a_i - b_i)^2 = \sum_{i=1}^{n} (\frac{B^2}{A^2} a_i^2 - 2\frac{B}{A} a_i b_i + b_i^2) = \frac{B^2}{A} - 2\frac{B^2}{A} + \sum_{i=1}^{n} b_i^2.$$

Rearranging this gives

$$B^2 \leq A \sum_{i=1}^{n} b_i^2.$$

This becomes

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$$

upon substituting in for A and B. □

In particular, if we set $b_1, ..., b_n = 1$, we have

$$(\sum_{i=1}^{n} a_i)^2 \leq n(\sum_{i=1}^{n} a_i^2).$$

This fact is used to prove the next lemma, which is somewhat of an improvement to the Cauchy-Schwarz inequality.

(2.1.6) Lemma. *For positive real numbers $x_1, x_2, ..., x_l$, let*

$$S = \frac{1}{l}\sum_{i=1}^{l} x_i$$

and, for some $j < l$,

$$s = \frac{1}{j}\sum_{i=1}^{j} x_i.$$

If $j \geq \alpha l$ and $|S - s| \geq \beta$, then

$$\frac{1}{l}\sum_{i=1}^{l} x_i^2 \geq S^2 + \alpha\beta^2.$$

Proof. Let $j \geq \alpha l$ be given, and

$$s = \frac{1}{j}\sum_{i=1}^{j} x_i.$$

satisfy $|S - s| \geq \beta$. With

$$T_j = \frac{1}{l-j}\sum_{i=j+1}^{l} x_i = \frac{lS - js}{l-j},$$

and the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{l} x_i^2 &\geq \frac{1}{j}(\sum_{i=1}^{j} x_i)^2 + \frac{1}{l-j}(\sum_{i=j+1}^{l} x_i)^2 \\
&= j(\frac{1}{j}\sum_{i=1}^{j} x_i)^2 + (l-j)(\frac{1}{(l-j)}\sum_{i=j+1}^{l} x_i)^2 \\
&= js^2 + (l-j)T_j^2
\end{aligned}$$

Then

$$js^2 + (l-j)T_j^2 = js^2 + \frac{(lS - js)^2}{l-j} = \frac{jls^2 - (js)^2 + (lS)^2 - 2jlsS + (js)^2}{l-j},$$

or

$$\sum_{i=1}^{l} x_i^2 \geq \frac{jls^2 + (lS)^2 - 2jlsS}{l-j} = lS^2 + \frac{jls^2 + (lS)^2 - 2jlsS - l^2S^2 - ljS^2}{l-j}$$

or

$$\sum_{i=1}^{l} x_i^2 \geq lS^2 + \frac{jls^2 - 2jlsS - ljS^2}{l-j} = lS^2 + \frac{lj}{l-j}(S-s)^2 \geq lS^2 + l\alpha\beta^2.$$

Dividing through by l gives the result. □

2.2 Statement and Proof of the Regularity Lemma

We are now in a position to state and prove a version of the Regularity Lemma. The proof follows that given by Bollobas in [2].

(2.2.1) Lemma. *Let $0 < \epsilon < \frac{1}{2}$ be given, as well as positive integers k and L. There exist integers M and N such that given any set G_1, G_2, ..., G_L of graphs on $V_k(n)$, where $n \geq N$, there exists a partition $\mathcal{P} = \{X_{i,j}\}_{i=1;j=1}^{m;k} \bigcup \{X_0\}$ of $V_k(n)$ such that $k \leq m \leq M$, X_0 is the exceptional class, $X_{i,j} \subseteq V_j$, and $\mathcal{P}$ is (ϵ, L)-uniform on each G_i.*

Proof. Assume the statement is false, and let $0 < \epsilon < \frac{1}{2}$, k, L be given. We let $f_1(x) = 4^x - 2^x$, and $f_r(x) = f(f_{r-1}(x))$. Pick $\alpha(0)$ to be the minimum integer such that $k2^{1-\alpha(0)} \leq \frac{\epsilon^5}{8}$ and $\frac{k}{\alpha(0)} < \frac{\epsilon}{2}$. Define the sequence

$$\{\alpha(i) = f_i(\alpha(0))\}_{i=1}^{T},$$

where we set $T = \lfloor \frac{4}{\epsilon^5} \rfloor + 1$. Also, define the sequence

$$\beta_i = \prod_{j=0}^{i} \alpha(j).$$

Let

$$n \geq N = \max\{\alpha(0)4^{\sum_{i=0}^{T} \alpha(i) + \alpha(T)}, \frac{\beta_T}{1-\epsilon}\},$$

$M = \beta_{T-1}$, and let graphs $G_1, G_2, ..., G_L$ on $V_k(n)$ be given. Start with a partition of $V_k(n)$ given by

$$\mathcal{P}_0 = \{X_{i,j}\}_{i=1,j=1}^{\beta_0,k} \bigcup \{X_0\},$$

where $|X_0| \leq \frac{\epsilon n}{2}$, and $|X_{i,j}| = t_0 = \lfloor \frac{n}{\alpha(0)} \rfloor$. This is possible, as $n > \alpha(0)$, and for $|X_0|$ we have at most $t_0 k \leq \frac{nk}{\alpha(0)} < \frac{\epsilon n}{2}$. By assumption, this partition is not (ϵ, L)-uniform so we need to refine our partition.

Consider a pair $(X_{i,j}, X_{p,q})$, $j \neq q$, that is not ϵ-regular in G_l. Then there exist sets $Y_{(i,j),(p,q),l} \subseteq X_{i,j}$, $Y_{(p,q),(i,j),l} \subseteq X_{p,q}$, such that $|Y_{(i,j),(p,q),l}| \geq \epsilon |X_{i,j}|$, $|Y_{(p,q),(i,j),l}| \geq \epsilon |X_{p,q}|$ and

$$|d_l(X_{i,j}, X_{p,q}) - d_l(Y_{(i,j),(p,q),l}, Y_{(p,q),(i,j),l})| > \epsilon,$$

where d_l is the density in the graph G_l. For a pair $(X_{i,j}, X_{p,q})$ that is ϵ-regular in G_l, let $Y_{(p,q),(i,j),l} = Y_{(i,j),(p,q),l} = \emptyset$. So, for any particular $X_{(i,j)}$, for each l we have sets

$$Y_{(i,j)(1,1),l}, ..., Y_{(i,j)(\beta_0,1),l}, Y_{(i,j)(1,2),l}, ..., Y_{(i,j)(\beta_0,2),l}, ...,$$

$$Y_{(i,j)(\beta_0,j-1),l}, Y_{(i,j),(1,j+1),l}, ..., Y_{(i,j),(\beta_0,k),l}.$$

This gives a total of $\beta_0(k-1)L$ sets.

There are a total of $2^{\beta_0(k-1)L}$ combinations of these sets. This defines equivalence classes for the vertices of $X_{(i,j)}$ according to the maximal combination that a particular vertex belongs to. We partition $X_{(i,j)}$ into $\alpha(1)$ sets by dividing each equivalence class into sets of size

$$t_1 = \lfloor \frac{t_0}{4^{\alpha(0)}} \rfloor,$$

where

$$\alpha(1) = 4^{\alpha(0)} - 2^{\alpha(0)}.$$

Any extra members are added to X_0.

The index of the initial partition is defined to be

$$I(\mathcal{P}_0) = \frac{1}{L\binom{k}{2}\beta_0^2} \sum_{l=1}^{L} \sum_{1\le i<j\le k} \sum_{p,q=1}^{\beta_0} d_l^2(X_{p,i}, X_{q,j})),$$

and $0 \le I(\mathcal{P}_0) \le 1$. We now have a new partition of $V_k(n)$, say

$$\mathcal{P}_1 = \{Z_0\} \bigcup \{Z_{i,j}\}_{i=1,j=1}^{\beta_1,k},$$

where $\beta_1 = \alpha(1)\alpha(0)$. Our goal is to show that

$$I(\mathcal{P}_1) = \frac{1}{L\binom{k}{2}\beta_1^2} \sum_{l=1}^{L} \sum_{1\le i<j\le k} \sum_{p,q=1}^{\beta_1} d_l^2(Z_{p,i}, Z_{q,j}) \ge I(\mathcal{P}_0) + C$$

for some positive number C that is independent of the partition, i.e., C depends on the values of ϵ, k, and L.

We begin by defining

$$\overline{X}_{i,j} = \bigcup \{Z_{r,j} : Z_{r,j} \subseteq X_{i,j}\},$$

and

$$\overline{X}_{(i,j),(p,q),l} = \bigcup \{Z_{r,j} : Z_{r,j} \subseteq X_{(i,j),(p,q),l}\}.$$

We have that

$$|X_{i,j} - \overline{X}_{i,j}| = t_0 - Jt_1 \le t_0 - (4^{\alpha(0)} - 2^{\alpha(0)})(\frac{t_0}{4^{\alpha(0)}} - 1) \le \frac{t_0}{2^{\alpha(0)}} + 4^{\alpha(0)} \le \frac{\epsilon^5 t_0}{8}.$$

This follows because we have

$$4^{\alpha(0)} \le 4^{\sum_{i=0}^{T} \alpha(i) + \alpha(T)} \le \frac{n}{\alpha(0)},$$

and so

$$4^{\sum_{i=0}^{T}\alpha(i)+\alpha(T)} \leq t_0$$

because the left hand side is an integer. Then

$$4^{\alpha(0)} \leq \frac{t_0}{4^{\alpha(0)}} < \frac{t_0}{2^{\alpha(0)}}$$

Finally, we have chosen

$$2^{1-\alpha(0)} < \frac{\epsilon^5}{8}.$$

Lemma **??** then gives, for each l,

$$|d_l(X_{i,j}, X_{p,q}) - d_l(\overline{X}_{i,j}, \overline{X}_{p,q})| < \frac{\epsilon^5}{4},$$

and

$$|d_l^2(X_{i,j}, X_{p,q}) - d_l^2(\overline{X}_{i,j}, \overline{X}_{p,q})| < \frac{\epsilon^5}{2}.$$

It then follows that

$$\frac{1}{L\binom{k}{2}\beta_0^2}\sum_{l=1}^{L}\sum_{1\leq i<j\leq k}\sum_{p,q=1}^{\beta_0} d_l^2(\overline{X}_{p,i}, \overline{X}_{q,j)}) \geq I(\mathcal{P}_0) - \frac{\epsilon^5}{2}.$$

For any pair $(X_{i,j}, X_{p,q})$ and each l, the Cauchy-Schwarz inequality gives

$$\frac{1}{\alpha(1)^2}\sum_{Z_{r,j}\subseteq X_{i,j}}\sum_{Z_{s,q}\subseteq X_{s,q}} d_l^2(Z_{r,j}, Z_{s,q}) \geq [\frac{1}{\alpha(1)^2}\sum_{Z_{r,j}\subseteq X_{i,j}}\sum_{Z_{s,q}\subseteq X_{s,q}} d_l(Z_{r,j}, Z_{p,q})]^2,$$

and

$$[\frac{1}{\alpha(1)^2}\sum_{Z_{r,j}\subseteq X_{i,j}}\sum_{Z_{s,q}\subseteq X_{s,q}} d_l(Z_{r,j}, Z_{p,q})]^2 = \frac{e^2(\overline{X}_{i,j}, \overline{X}_{p,q})}{(\alpha(1)t_1)^2} = d_l^2(\overline{X}_{i,j}, \overline{X}_{p,q}),$$

so that

$$\frac{1}{\alpha(1)^2}\sum_{Z_{r,j}\subseteq X_{i,j}}\sum_{Z_{s,q}\subseteq X_{s,q}} d_l^2(Z_{r,j},Z_{p,q}) \geq d_l^2(\overline{X}_{i,j},\overline{X}_{p,q}).$$

Assume that some pair $(X_{i,j},X_{p,q})$ is not ϵ-regular in G_l, and consider the set $\overline{X}_{(i,j),(p,q),l}$. We have that $|X_{(i,j),(p,q),l}| \geq \epsilon|X_{i,j}|$, so that

$$|X_{(i,j),(p,q),l} - \overline{X}_{(i,j),(p,q),l}| \leq |X_{i,j} - \overline{X}_{i,j}| \leq \frac{\epsilon^5 t_0}{8} \leq \frac{|X_{(i,j),(p,q),l}|\epsilon^4}{8}.$$

Lemma **??** now gives

$$|d_l(X_{(i,j),(p,q),l}, X_{(p,q),(i,j),l}) - d_l(\overline{X}_{(i,j),(p,q),l}\overline{X}_{(p,q),(i,j),l})| \leq \frac{\epsilon^4}{4},$$

Also

$$|\overline{X}_{(i,j),(p,q),l}| \geq |X_{(i,j),(p,q),l}| - |X_{(i,j)} - \overline{X}_{(i,j)}| \geq (\epsilon - \frac{\epsilon}{\alpha(0)2^{\alpha(0)}})|\overline{X}_{i,j}| \geq (1-2^{-7})\epsilon|\overline{X}_{i,j}|.$$

A simple use of the triangle inequality gives

$$|d_l(\overline{X}_{i,j},\overline{X}_{p,q}) - d_l(\overline{X}_{(i,j),(p,q),l}\overline{X}_{(p,q),(i,j),l})| \geq \epsilon - \frac{\epsilon^4}{4} - \frac{\epsilon^5}{4} \geq \frac{15}{16}\epsilon.$$

We let

$$S = d_l(\overline{X}_{i,j},\overline{X}_{p,q}) = \frac{1}{\alpha(1)^2}\sum_{Z_{r,j}\subseteq X_{i,j}}\sum_{Z_{p,q}\subseteq X_{s,q}} d_l(Z_{r,j},Z_{p,q}),$$

and

$$s = d_l(\overline{X}_{(i,j),(p,q),l},\overline{X}_{(p,q),(i,j),l}) = \frac{1}{A_1A_2}\sum_{Z_{r,j}\subseteq \overline{X}_{(i,j),(p,q),l}}\sum_{Z_{u,q}\subseteq \overline{X}_{(p,q),(i,j),l}} d_l(Z_{r,j},Z_{u,q}),$$

where A_1 is the number of sets $Z_{r,j} \subseteq \overline{X}_{(i,j),(p,q),l}$, and A_2 is the number of sets $Z_{s,q} \subseteq \overline{X}_{(p,q),(i,j),l}$. We have that

$$\frac{A_1 A_2}{\alpha(1)^2} = \frac{|\overline{X}_{(i,j),(p,q),l}||\overline{X}_{(p,q),(i,j),l}|}{|\overline{X}_{i,j}|, |\overline{X}_{p,q}|}$$

With the fact that

$$|S - s| \geq \frac{15}{16}\epsilon,$$

Lemma **??** gives

$$\frac{1}{\alpha(1)^2} \sum_{Z_{r,j} \subseteq X_{i,j}} \sum_{Z_{s,q} \subseteq X_{s,q}} d_l^2(Z_{r,j}, Z_{s,q}) \geq d_l^2(\overline{X}_{i,j}, \overline{X}_{p,q}) + \frac{|X_{(i,j),(p,q),l}||X_{(p,q),(i,j),l}|}{|\overline{X}_{i,j}||\overline{X}_{p,q}|} \frac{15^2}{16^2}\epsilon^2,$$

or

$$\frac{1}{\alpha(1)^2} \sum_{Z_{r,j} \subseteq X_{i,j}} \sum_{Z_{s,q} \subseteq X_{s,q}} d_l^2(Z_{r,j}, Z_{p,q}) \geq d_l^2(\overline{X}_{i,j}, \overline{X}_{s,q}) + \frac{3}{4}\epsilon^4$$

Then we have that

$$I(\mathcal{P}_1) = \frac{1}{L\binom{k}{2}(\beta_1)^2} \sum_{l=1}^{L} \sum_{1 \leq i < j \leq k} \sum_{p,q=1}^{\beta_1} d_l^2(Z_{p,i}, Z_{q,j})$$

satisfies

$$I(\mathcal{P}_1) \geq \frac{1}{L\binom{k}{2}\beta_0^2} \sum_{l=1}^{L} \sum_{1 \leq i < j \leq k} \sum_{p,q=1}^{\beta_0} d_l^2(\overline{X}_{p,i}, \overline{X}_{q,j)}) + \frac{3}{4}\epsilon^5,$$

as at least $\epsilon L\binom{k}{2}\beta_0^2$ pairs are not ϵ-regular. Which gives the desired result, namely,

$$I(\mathcal{P}_1) \geq I(\mathcal{P}_0) - \frac{\epsilon^5}{2} + \frac{3}{4}\epsilon^4 \geq I(\mathcal{P}_0) + \frac{\epsilon^5}{4}.$$

Moreover, we have that

$$|Z_0| \leq |X_0| + \beta_0 k \frac{t_0}{2^{\alpha(0)}} \leq |X_0| + \frac{kn}{2^{\alpha(0)}} < \epsilon n.$$

We now generate a sequence of partitions $\mathcal{P}_0, \mathcal{P}_1, ..., \mathcal{P}_T$, none of which, by assumption, is (ϵ, L)-regular on all L graphs. We note that all partitions exist, as $n > \frac{\beta_T}{1-\epsilon}$ and for Z_0 in $\mathcal{P}_i$, we have, for every $i \leq T$, that

$$|Z_0| \leq |X_0| + \sum_{j=0}^{i} \frac{kn}{2^{\alpha(j)}} \leq \frac{\epsilon n}{2} + \frac{kn}{2^{\alpha(0)-1}} < \epsilon n.$$

Moreover, we have that

$$I(\mathcal{P}_i) \geq I(\mathcal{P}_{i-1}) + \frac{\epsilon^5}{4}$$

because

$$4^{\alpha(i)} \leq \frac{t_i}{2^{\alpha(i)}},$$

which is proven similarly to the case when $i = 1$.

Thus, as none of the partitions is (ϵ, L)-regular, we have that

$$I(\mathcal{P}_T) > T\frac{\epsilon^5}{4} > 1.$$

This is a contradiction, and so one of our partitions $\mathcal{P}_0, \mathcal{P}_1, ..., \mathcal{P}_{T-1}$ must be (ϵ, L)-regular on all L graphs, and each partite class is partitioned into at most M sets. This proves the result. $\square$

Chapter 3

Embedding Lemmas

3.1 Bounded Degree Graphs

One of the most well known graph embedding lemmas is called the Blow-Up Lemma. We do not attempt to provide a proof, but the lemma is as follows:

(3.1.1) Lemma. *Given a graph G of size g, and positive parameters δ, Δ, there exists a positive number $\epsilon = \epsilon(\delta, \Delta, g)$ such that the following holds: For arbitrary positive integers $n_1, n_2, ..., n_g$, replace the vertices $v_1, v_2, ..., v_g$ of G with disjoint vertex sets $V_1, V_2, ..., V_g$ of sizes $n_1, n_2, ..., n_g$. We create two graphs, the first, H, is obtained by replacing each edge (v_i, v_j) of G by the complete bipartite graph for (V_i, V_j). The second graph R is obtained by replacing each edge (v_i, v_j) of G by an arbitrary (ϵ, δ)-super-regular pair between (V_i, V_j). Any graph S with $\Delta(S) \leq \Delta$ that is embeddable into H is also embeddable into R.*

For proof, the reader can see [8], where the result is proven using a randomized algorithm.

The main goal of this chapter is to consider embedding more general classes of graphs than just those of bounded degree by using the Blow-Up Lemma as a loose model. To do

this, we introduce a definition. This provides us with the ability to make more general statements later on without specifying a particular class of graphs.

(3.1.2) Definition. Let $\mathcal{G}$ be a collection of graphs, and $f : \mathcal{G} \times (0,1) \to \mathbb{N}$. If, for every $\delta \in (0,1)$, there exists $\epsilon > 0$ such that any $G \in \mathcal{G}$ is embeddable into any (ϵ, δ)-super-regular graph on $V_{\chi(G)}(f(|G|, \delta))$, then $\mathcal{G}$ is said to be (f, δ)-embeddable.

It is only a small step to relax the Blow-Up Lemma to a statement in terms of this idea. However, as we see in this chapter, many more classes of graphs can be studied with this definition because we have the ability to vary the function f. To go so far as to loosen the statement of the Blow-Up Lemma, something we are about to do, we in fact need one small result. To make this result more general, we introduce a new class of graphs.

(3.1.3) Definition. If G is a graph with n vertices such that there exists a sequence of the vertices of G, say $v_1, v_2, ..., v_n$, such that

$$N_L(v_i) = \{v_j : v_i v_j \in E(G), j < i\},$$

satisfies $|N_L(v_i)| \leq d$, then G is d-degenerate.

For a given d, we set $\mathcal{D}_d$ to be the class of d-degenerate graphs. If $\mathcal{B}_d$ is the class of graphs with maximum degree at most d, then it is easily seen that $\mathcal{B}_d \subseteq \mathcal{D}_d$. Hence the following result holds for $\mathcal{B}_d$ as well.

(3.1.4) Lemma. *Let $H \in \mathcal{D}_d$. We have that $\chi(H) \leq d + 1$.*

Proof. If G is any graph with at most $d+1$ vertices, then the result is trivially true. Now assume that the result is true with for all graphs $G \in \mathcal{D}_d$ with $|G| \leq n$, and let $H \in \mathcal{D}_d$ with $|H| = n + 1$. Then there exists a sequence of the vertices of H, say $v_1, ..., v_n, v_{n+1}$,

that proves that H is d-degenerate. The induced subgraph from the vertices $v_1, ..., v_n$, say H', is also d-degenerate, so that the induction hypothesis gives us that H' is embeddable into $K_{n;d+1}$. As v_{n+1} has at most d neighbors in H, there exists a color class, say V_α, that contains no neighbors of v_{n+1}. Thus, any selection of a vertex not previous mapped to in this color class for the image of v_{n+1} suffices as we are embedding into a complete graph. Because there are $n+1$ vertices in each color class, this is guaranteed to work. □

There is a stronger version of Lemma **??**, stated in terms of maximum degree, known as Brooks' Lemma. More specifically:

(3.1.5) Lemma. *If G is a connected graph that is not a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

It is now simply a small step to ensure that the Blow-Up Lemma actually says what was stated above.

(3.1.6) Lemma. *For each Δ, $\mathcal{B}_\Delta$ is $(|G|, \delta)$-embeddable.*

Proof. We let $\delta \in (0,1)$ be given, as well as Δ. For each $i = 1, 2, ..., \Delta + 1$, the Blow-Up Lemma asserts that we have an ϵ_i such that each graph $G \in \mathcal{B}_\Delta$ with $\chi(G) = i$ is embeddable into any (ϵ_i, δ)-super-regular graph on $V_i(|G|)$, as G is easily embeddable into $K_{|G|,i}$. This covers all $G \in \mathcal{B}_\Delta$ by lemma **??**. Thus, for any $\delta \in (0,1)$, we select

$$\epsilon = \min\{\epsilon_i, 1 \leq i \leq \Delta + 1\}.$$

This completes the proof. □

3.2 Complete Multipartite Graphs

We begin this section by defining the class of graphs $\mathcal{C}_{k_0} = \{K_{n,k} : k \leq k_0\}$. Studying this class of graphs gives us, in the opinion of the author, the most interesting result of this chapter, Corollary **??**. To achieve this, we need to employ the following Lemmas. The first, stated without proof, is due to Erdős. The second guarantees that (ϵ, δ)-super-regular graphs on $V_k(n)$ have many copies of K_k if ϵ is small.

(3.2.1) Lemma. *Let A_i be given for $i = 1, ..., k$, $|A_i| = N$ for each i, and let $m \in \mathbb{N}$ be given. If $G \subseteq A_1 \times ... \times A_k$ and $|G| \geq N^{k-1/m^{(k-1)}}$, then there exist sets $B_i \subseteq A_i$ such that $|B_i| = m$ for all i and $B_1 \times ... \times B_k \subseteq G$.*

(3.2.2) Lemma. *Let $k \geq 2$ and $\delta > 0$ be given. There exists an $\epsilon > 0$ and a function $g_k(\delta)$ such that any graph H that is (ϵ, δ)-super-regular on $V_k(N)$ contains at least $g_k(\delta)N^k$ subgraphs isomorphic to K_k.*

Proof. The result is shown by induction on k. Let $k = 2$, and H be any (ϵ, δ)-super-regular graph on $V_2(N)$ where ϵ is arbitrary. Each vertex in V_1 has at least δN neighbors in V_2, so that there are at least δN^2 edges in H. This proves the case $k = 2$.

Now assume that the result is true for $k = k_0$. Let δ be given. Denote by ϵ' and $g_{k_0}(\delta/2)$ the parameters guaranteed by the induction for the choice of k_0 and $\delta/2$. Pick

$$\epsilon = \min\{\frac{\epsilon'\delta}{2}, \frac{\delta}{2(k-2)}\},$$

and $g_{k_0+1}(\delta) = (\delta/2)^{k_0} g_{k_0}(\delta/2)$. Also, let H be any (ϵ, δ)-super-regular graph on $V_{k_0+1}(N)$.

Pick $v \in V_{k_0+1}$, and consider $N_i(v) = \{w : w \in V_i, vw \in E(H)\}$ for $i = 1, ...k_0$. It follows from the (ϵ, δ)-super-regularity of H that $|N_i(v)| \geq \delta N$ for each i. Let us randomly choose a set of $N_i'(v) \in N_i(v)$ of size $\lfloor \delta N \rfloor$ for each i. Denote the k_0-partite induced subgraph on $(N_1'(v), ..., N_{k_0}'(v))$ by H_1. For some i, in the pair $(N_i'(v), N_j'(v))$, at

most $\epsilon(\delta N)$ of the vertices of $N_i'(v)$ have less than $\delta(\delta N)$ neighbors in $N_j'(v)$. Thus, for this i, there are at most $\epsilon(k_0-1)N$ vertices in $N_i'(v)$ that have less than $\delta(\delta N)$ neighbors in all pairs $(N_i'(v), N_j'(v))$. Let $\bar{N}_i(v) \subseteq N_i'(v)$ be a collection of size $N^* = \lfloor(\delta-\epsilon(k_0-1))N\rfloor \geq (\delta/2)N$ such that each vertex in $\bar{N}_i(v)$ has at least $\delta(\delta N)$ neighbors in each $N_j(v)$, $j \neq i$. After doing so for each i, let the induced subgraph on $(\bar{N}_1(v), ..., \bar{N}_{k_0}(v))$ be given by H_2.

Now for any pair $(\bar{N}_i(v), \bar{N}_j(v))$, any vertex in $\bar{N}_i(v)$ has degree at least $\delta(N^*) - \epsilon(k_0-1)N^* \geq (\delta/2)(\delta N/2)$. Also, the condition $\epsilon N \leq \epsilon'((\delta/2)N)$ implies now that we have that H_2 is $(\epsilon', \delta/2)$-super-regular (isomorphically) on $V_{k_0}(N^*)$, where $N^* \geq (\delta/2)N$. Thus, by induction, we have that H_2 contains at least

$$g_{k_0}(\delta/2)N^{*k_0} \geq (\delta/2)^{k_0} g_{k_0}(\delta/2)N^{k_0} = g_{k_0+1}(\delta)N^{k_0}$$

copies of K_k.

To finish the proof, we sum over all possible vertices of $v \in V_{k_0+1}$ and obtain $g_{k_0+1}(\delta)N^{k_0+1}$ copies of K_{k_0+1} in H. Thus, induction gives the result for all k. □

Putting these two lemmas together gives us the following result for the class of complete multipartite graphs with bounded chromatic number.

(3.2.3) Lemma. *Given k_0, there exists a function $g(\delta)$ such that $\mathcal{C}_{k_0}$ is $(g(\delta)^{-|G|^{k_0-1}}, \delta)$-embeddable.*

Proof. Given k_0, let $K_{m;k} \in \mathcal{C}_{k_0}$ and also let δ be given. From Lemma **??**, there exists an A and an ϵ such that any (ϵ, δ)-super-regular graph on $V_{k'}(n)$ contains at least $An^{k'}$ copies of $K_{k'}$, where $2 \leq k' \leq k_0$.

With this ϵ and A, let H be any (ϵ, δ)-super-regular graph on $V_k(N)$, where $N \geq A^{-|K_{m;k}|^{k_0-1}} > A^{-m^{k-1}}$. We then have at least $AN^k > N^{k-1/m^{k-1}}$ copies of K_k in H. By appealing to Lemma **??**, we can then guarantee a copy of $K_{m,k}$ in H. □

We now have the previously mentioned corollary.

(3.2.4) Corollary. *Let $\mathcal{G}$ be a class of graphs with bounded chromatic number. There exists an f such that $\mathcal{G}$ is (f, δ)-embeddable.*

3.3 p-arrangeable Graphs

A slightly more interesting example of a class of graphs to study is the collection of p-arrangeable graphs. For some fixed p, define $\mathcal{P}_p$ to be the collection of graphs that are p-arrangeable. Similar to the class of graphs with bounded degree, we have that $\mathcal{P}_p \subseteq \mathcal{D}_p$, so that Lemma **??** again applies. Also similar to the class of bounded degree graphs, we have that $\mathcal{P}_p$ is (f, δ)-super-regular with f linear.

(3.3.1) Lemma. *For a fixed p, $\mathcal{P}_p$ is $(\lfloor 2\delta^{-p}|G| \rfloor + 1, \delta)$-embeddable.*

Proof. Let $\delta > 0$ and $p \geq 1$ be given. Also, let G be p-arrangeable with $|G| = n$ and $\chi(G) = k$. Set

$$\epsilon = \frac{\delta^p}{(p+1)2^p},$$

and, for convenience, $A = 2\delta^{-p}$. Finally, let H be any (ϵ, δ)-super-regular graph on $V_k(N)$, $N \geq An$.

We denote the collections of an assignment of the vertices of G into color classes of G by $V_1, V_2, ..., V_k$, and let $V_k(N) = \{W_1, W_2, ..., W_k\}$. We embedd the vertices of G into H according to a sequence $v_1, v_2, ..., v_n$ which proves that G is p-arrangeable. We map $h : V(G) \to V(H)$ such that

1. If $v_\alpha \in V_\beta$, then $h(v_\alpha) \in W_\beta$,

2. If $v_{\alpha_1} v_{\alpha_2} \in E(G)$, then $h(v_{\alpha_1})h(v_{\alpha_2}) \in E(H)$,

3. With $v_\alpha \in V_\beta$, $V(\alpha, i) = \{h(v_j) : 1 \leq j \leq i, v_i v_j \in E(G)\}$, and $x = |V(\alpha, i)|$, we have a subset $W'_\beta \subseteq W_\beta$ such that $|W'_\beta| \geq \delta^x |W_\beta| = \delta^x N$ and all vertices of W'_β are adjacent to all vertices of $V(\alpha, i)$.

To guarantee that this can be done, we proceed inductively. Let $v_1 \in V_\beta$. Any selection of a vertex in W_β as $h(v_1)$ obviously satisfies 1. and 2. It follows from the fact that all pairs $(W_\beta, W_{\beta'})$ are (ϵ, δ)-super-regular that any choice for $f(v_1)$ has degree at least δN in each pair, so that 3. is satisfied as well.

Now assume that the vertices $h(v_1), ..., h(v_i)$ have been selected and 1.-3. are satisfied. With $v_{i+1} \in V_\gamma$, by 3. we have a set W'_γ where each vertex is adjacent to all vertices of $V(i+1, i)$. Any selection of a vertex in W'_γ as $h(v_{i+1})$ satisfies 1. and 2.. However, we must use care in selecting $h(v_{i+1})$ to ensure that 3. is again satisfied. It is clear that we only need to consider $V(\alpha, i+1)$ when $v_\alpha v_{i+1} \in E(G)$ and $\alpha > i+1$.

Given a particular $\alpha > i+1$ with $v_\alpha \in V_\nu$ and $v_\alpha v_{i+1} \in E(G)$, consider $V(\alpha, i+1)$. We have the obvious fact that $|V(\alpha, i+1)| = 1 + |V(\alpha, i)|$. In W_ν there is a subset W'_ν of size $\delta^{x-1}N$, with $x = |V(\alpha, i+1)|$, that is adjacent to all vertices of $V(\alpha, i)$ of size $\delta^{x-1}N$. Because $x \leq p$, $\delta^x \geq \epsilon$, and so W'_ν and W'_γ are larger than ϵN. It follows that at most ϵN of the vertices of W'_γ are adjacent to less than $\delta |W'_\nu|$ of the vertices of W'_ν. Thus there exists a portion of the vertices of W'_γ that satisfy 3. for this choice of α.

Here is where we need to use the fact that G is p-arrangeable. We have that

$$|\bigcup_{v_\alpha \in N_R(v_{i+1})} N_{L_{i+1}}(v_\alpha)| \leq p.$$

Hence, for any v_α adjacent to v_{i+1}, $\alpha > i+1$, it follows then that $N_{L_{i+1}}(v_\alpha)$ is one of at most 2^{p-1} distinct possible subsets of

$$\bigcup_{v_\alpha \in N_R(v_{i+1})} N_{L_{i+1}}(v_\alpha),$$

noting that v_{i+1} is contained in each set. Any two right neighbors of v_{i+1} in the same color class, say v_α and v_β, with $N_{L_{i+1}}(v_\alpha) = N_{L_{i+1}}(v_\beta)$ have the same selection set. With k color classes, it then follows that there are at most $k2^{p-1}$ selection sets that contain all possible right neighbors of v_{i+1}. For any of these possible selection sets, there are more than $(1-\epsilon)|W'_\gamma|$ vertices of W'_γ that, if chosen for $h(v_i+1)$, have a suitable number of neighbors, namely δ percent, in reference to a particular selection set. To guarantee that 3. is again satisfied, note that $|W'_\gamma| \geq \delta^p N$, so we have at least

$$\delta^p N - k2^{p-1}\epsilon N = \frac{1}{2}\delta^p N \geq n$$

vertices of W'_γ that we can choose for $h(v_{i+1})$ that satisfy 3. Note that we have tacitly applied Lemma **??** to bound k. Because $|G| = n$, this guarantees that there is a choice for $h(v_{i+1})$ that has not been selected for any previous vertex. This process can continue by induction until all $h(v_j)$, $1 \leq j \leq n$, have been selected. □

3.4 Some Notes on d-degenerate Graphs

The d-degenerate graphs have been introduced to provide a nice way to bound chromatic numbers. However, it turns out that these are precisely the same graphs considered by the Burr-Erdős conjecture.

(3.4.1) Proposition. *G is a graph such that every subgraph H of G has $\delta(H) \leq d$ if and only if G is d-degenerate.*

Proof. We prove necessity by induction. If $|G| \leq d$, then this is trivial. Assume now that this is true for all G with $H \subseteq G$ implying $\delta(H) \leq d$ and $|G| \leq M$. Take G to be such a graph with $|G| = M+1$. Then G has at least one vertex of degree at most d. Select such a vertex as v_{M+1}. Then $G' = G - v_{M+1}$ has $|G'| = M$, and $H \subseteq G'$ implies

$H \subseteq G$, so that $H \subseteq G'$ implies $\delta(H) \leq d$. Therefore, the induction hypothesis implies the result for G', and thus the result is true for G with our choice of v_{M+1}. Induction then guarantees the result.

Now let G be d degenerate and $G = n$. Then we have a sequence of the vertices of G that proves that G is d-degenerate, say $v_1, ...v_n$. For any $H \subseteq G$, there exists a vertex of H with largest index in reference to the d-degenerate sequence, say v_α. Obviously, in H, $\deg(v_\alpha) \leq d$, so that $\delta(H) \leq d$. This proves the result. □

In reference to the embedding methods of this chapter, the idea of d-degenerate is much more informative than the subset criterion. d-degenerate graphs however elude the concept of (f, δ)-embeddability without appealing directly to Corollary **??**. To be more general about what we can do, let us extend the idea of degenerate graphs.

(3.4.2) Definition. Let $f : \mathbb{N} \to \mathbb{N}$ be an non-decreasing function. A graph G is f-degenerate if there exists a sequence of the vertices of G, say $v_1, v_2, ..., v_n$, such that, for all i,

$$|N_L(v_i)| \leq f(i).$$

We shall denote the class of f-degenerate graphs by $\mathcal{D}_f$, and, more importantly, we let $\mathcal{D}_{f,k_0} = \{G : G \in \mathcal{D}_f, \chi(G) \leq k_0\}$.

(3.4.3) Proposition. *Let $G \in \mathcal{D}_{f,k_0}$. Define*

$$\zeta(n) = \sum_{j=1}^{f(n)} \binom{n}{j}.$$

If $0 < \delta < 1$ is given, then for any

$$\epsilon \leq \frac{\delta^{f(n)}}{2k_0\zeta(n)},$$

we have that G is embeddable into any (ϵ, δ)-super-regular graph on $V_{\chi(G)}(\lfloor 2\delta^{f(n)} n \rfloor + 1)$.

We do not explicitly prove this result. To actually do so is essentially to provide a reproduction of the proof of Lemma **??**. Recovering the case for d-degenerate graphs amounts to setting $f(i) = d$ for all i. In this particular case, we do see that the size of the color classes does grow linearly with $|G|$, and the extra bounding of the chromatic number is in fact unnecessary. On the other hand, in this case it is true this result implies that must let ϵ decreases as a polynomial in $|G|$. It is this inability to fix ϵ that results in the failure of this method to prove the Burr-Erdős conjecture even though Corollary **??** guarantees that in some we can actually do so.

Chapter 4

Ramsey's Theorem on Multipartite Hosts

4.1 The Theorem and the Proof

The goal here is to color complete multipartite graphs and search for forced monochromatic subgraphs. However, some care needs to be taken when we are looking for some prescribed subgraph. Consider a graph G with chromatic number k. If we 2-color $K_{N,k'}$, can we guarantee a monochromatic copy of G if N is sufficiently large? A negative answer is easily given if $k' < r(k)$. To see this, let χ be a two coloring $K_{k'}$ that admits no monochromatic copy of K_k, which exists as $k' < r(k)$. We induce a new coloring on $K_{N,k'}$ by mapping the color classes bijectively to the vertices of K'_k by the use of some function f. We induce χ' on $K_{N,k'}$ by coloring an edge vw, $v \in V_\alpha$, $w \in V_\beta$, by setting $\chi'(vw) = \chi(f^{-1}(V_\alpha)f^{-1}(V_\beta))$. Then χ' admits no monochromatic graph with chromatic number at least k, hence no G. This motivates us to make the following definition.

(4.1.1) Definition. For a set of graphs $G_1, ..., G_m$, let $r = r_m(\chi(G_1), ..., \chi(G_m))$. The multipartite Ramsey number $p_m(G_1, ..., G_m)$ is defined be the minimal integer N such that any m-coloring of $K_{N,r}$ contains a monochromatic copy of G_i in color i, for some i.

This section is dedicated to proving that this definition makes sense, i.e., that the number p_m exists. Before we present the proof, we need to prove the following:

(4.1.2) Lemma. *Let G be a graph on $V_k(n)$ and $e(G) > (\binom{k}{2} - 1)n^2$. Then G contains a copy of K_k.*

Proof. In $K_{n,k}$, we have n^k different copies of K_k. Each edge of $K_{n,k}$ belongs to n^{k-2} of these copies. Thus, removing a single edge from $K_{n,k}$ destroys at most n^{k-2} copies of K_k, and to destroy all copies of K_k, we must remove at least $\frac{n^k}{n^{k-2}} = n^2$ edges. As $K_{n,k}$ has $\binom{k}{2}n^2$ edges, any graph on $V_k(n)$ with more than $(\binom{k}{2} - 1)n^2$ edges must contain a copy of K_k. □

Now with the aid of the Regularity Lemma and the Blow-Up Lemma we can prove what could be considered our main theorem.

(4.1.3) Theorem. *Given graphs $G_1, G_2, ..., G_c$, there exists an integer N such that any c-coloring of $K_{N;r}$, $r = r(\chi(G_1), \chi(G_2), ..., \chi(G_c))$, contains a monochromatic copy of G_i in color i for some $1 \leq i \leq m$.*

Proof. We let $\delta = \frac{1}{4c}$. Define $n = \max\{|G_i|\}$, $k = \max\{\chi(G_i)\}$, $\Delta = (k-1)n$, and again denote the standard Ramsey number $r(\chi(G_1), \chi(G_2), ..., \chi(G_c))$ simply by r. Also, we assume that $\chi(G_i) \geq 2$ for each i. By the Blow-Up Lemma, there exists an $\epsilon_i > 0$, $i = 2, 3, ..., k$, such that any (ϵ_i, δ)-super-regular graph on $V_{\chi(G_i)}(m)$, $m \geq n$, contains a copy of G_i. This follows because $\Delta(G_i) \leq \Delta$ for all i and each G_i is a subgraph of $K_{n;\chi(G_i)}$.

Pick

$$\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}, ..., \frac{\epsilon_c}{2}, \frac{1}{2cr^2}, \frac{1}{4kc}\}.$$

This gives that $0 < \epsilon < \frac{1}{2}$. We pick now N such that the Regularity Lemma with c graphs on $V_r(N)$ has an (ϵ, c)-uniform partition with sets of size S with $(1-\epsilon k)S \geq n$. We do not care about the exceptional class and ignore it.

Let $\chi : E(K_{N;r}) \to \{1, 2, ..., c\}$ be an arbitrary coloring, and define the graphs $H_j = (V_r(N), \chi^{-1}(j))$. We apply the Regularity Lemma with ϵ and c for this set of graphs. Let the guaranteed (ϵ, c)-uniform partition of the vertex set of $V_r(N)$ be given by the disjoint vertex sets $A_{i,j}$, where $A_{i,j}$ is the i'th set in the j'th partite class, $1 \leq i \leq \alpha$, $1 \leq j \leq r$. Define an auxillary graph H on $V_r(\alpha)$ by letting the vertex pair $(v_{i,j}, v_{i',j'})$ $(j \neq j')$ be an edge if the pair of sets $(A_{i,j}, A_{i',j'})$ is ϵ-regular. We have at most $\epsilon c(r\alpha)^2 < \alpha^2$ of the pairs not ϵ-regular, so that Lemma **??** guarantees that H contains a copy of K_r.

On this copy of K_r in H, we define a coloring based on the color densities of the pairs in the (ϵ, c)-uniform partition. Let the partitioned vertex sets corresponding to this copy of K_r be labeled as $X_1, ..., X_r$. Under χ, there are c induced graphs, one for each color, on each bipartite pair (X_i, X_j) with $i \neq j$. The sum of the densities of all c graphs adds to 1, so that there exists at least one with density at least $\frac{1}{c}$. We pick a color with density at least $\frac{1}{c}$, say from the graph G_j, and color the corresponding edge of our copy of K_r with color j. This gives a c-coloring of K_r, and hence, from Ramsey's theorem, there exists a monochromatic copy of $K_{\chi(G_l)}$ in color l in our copy of K_r.

For each of the vertices of our monochromatic $K_{\chi(G_l)}$, denote the associated vertex sets as $A_1, A_2, ..., A_{\chi(G_l)}$. Each pair is ϵ-regular with density at least $\frac{1}{c}$ in some color. For any given A_i, less than $\epsilon|A_i|$ vertices have degree less than $(\frac{1}{c} - \epsilon)|A_i| \geq \frac{|A_i|}{2c}$. Thus, at most $\epsilon(\chi(G_l) - 1)$ vertices exist in A_i with degree less than $\frac{1}{2c}|A_i|$ in all pairs (A_i, A_j), $j \neq i$. Denote by Y_i the set A_i minus these low degree vertices, with some extra vertices removed, if needed, to make $|Y_i| = |Y_j|$ for each $1 \leq i, j \leq \chi(G_l)$. We have that

$1 - \epsilon(\chi(G_l) - 1) > \frac{1}{2}$, so that for all $|B| \leq 2\epsilon|Y_i|$, $|C| \leq 2\epsilon|Y_j|$, we have $|B| \geq \epsilon|A_i|$, $|C| \geq \epsilon|A_j|$, and

$$|d(B,C) - d(Y_i,Y_j)| < |d(B,C) - d(A_i,A_j)| + |d(A_i,A_j) - d(Y_i,Y_j)| \leq 2\epsilon.$$

Thus, each pair (Y_i, Y_j) is 2ϵ-regular. Also

$$d(B,C) \geq d(A_i,A_j) - \epsilon \geq \frac{1}{c} - \frac{1}{4kc} > \frac{1}{4c} = \delta.$$

Moreover, $\epsilon(\chi(G_l) - 1) < \frac{1}{4c}$ implies that in each pair any vertex has degree at least $(\frac{1}{2c})|A_i| - \epsilon(\chi(G_l) - 1)|A_i| \geq \frac{1}{4c}|Y_i| = \delta|Y_i|$.

Thus, each pair (Y_i, Y_j) is $(2\epsilon, \delta)$-super-regular, and therefore also (ϵ_l, δ)-super-regular since $2\epsilon \leq \epsilon_l$, as if some vertex set is (ϵ, δ)-super-regular, then it is (ϵ', δ)-super regular for $\epsilon' \geq \epsilon$. Because $|Y_i| > (1 - \epsilon(\chi(G_l) - 1))|A_i| \geq (1 - \epsilon k)S \geq n$, the Blow-Up Lemma then guarantees that the graph on $Y_1, ..., Y_{\chi(G_l)}$ contains a copy of G_l. This proves the theorem. □

4.2 Bounds for (ϵ, δ)-embeddable Classes of Graphs

The proof of Theorem **??** provides no effective upper bound for the integer p_m. Of course it is desirable to know precisely what this number is, or at least have an effective method to calculate the precise number, for any given set of graphs. This is, as is the case with the regular Ramsey numbers, something that is well beyond our reach. However, we can pick apart the proof of Theorem **??** and use the results of the last chapter to yield a method that provides more reasonable bounds. The first thing that we can take is the following:

(4.2.1) Lemma. *Let $c > 1$ and a set of natural numbers $a_1, a_2, ..., a_c$ be given, as well as $n \in \mathbb{N}$. With $\gamma = \max\{a_i\}$ and $r = r_c(a_1, a_2, ..., a_c)$, we have that for $\delta = \frac{1}{4c}$ and any given $\epsilon > 0$ sufficiently small, there exists an A such that any c-coloring on $K_{An,r}$ contains, for at least one i, a monochromatic graph that is $(2\epsilon, \delta)$-super-regular on $V_{a_i}(N)$ where $n \geq N$.*

Proof. The proof is similar to the proof of Theorem **??**, and note that any $\epsilon < \frac{1}{4\gamma c}$ is sufficiently small.

Let $c > 1$, $a_1, a_2, ..., a_c$, and ϵ be given. We apply the Regularity Lemma with ϵ, $k = r(a_1, a_2, ..., a_c)$, and $L = c$ to give an M and an N such that any c graphs on $V_k(m)$ have an (ϵ, L)-uniform partition when $m > N$. Now set $A = \frac{M}{(1-\gamma\epsilon)}$.

Take any c-coloring, say χ, on $K_{An,k}$, and let $E(G_i) = \chi^{-1}(i)$ for $i = 1, ..., c$. We have a partition of $V_k(An)$ that is (ϵ, c) uniform on each G_i. Ignoring the exceptional class, the size of each of the sets in the partition is at least n. As in Theorem **??**, we end up with a monochromatic graph that is $V_{a_i}(s)$, for some i, where $s > \frac{An}{M}$.

This graph is ϵ-regular in each pair, and the density is at least $1/c$. We have at most ϵs of the vertices in each pair with degree less than $1/2c$. There are $\gamma = \max\{a_i\}$ pairs, so we lose at most $\gamma\epsilon s$ vertices in each partite class, and keep those that have large enough degree. To make this nice, we keep equivalent numbers of vertices in each class. To guarantee that this works, any vertex that we have kept in some pair loses at most $\gamma\epsilon$ percent of its degree, so each vertex has degree at least $1/(2c) - \gamma\epsilon > 1/(4c) = \delta$.

Since we keep more than $(1 - \gamma\epsilon)s \geq (1 - \gamma\epsilon)An/M$ vertices, the result follows with $A = \frac{M}{1-\gamma\epsilon}$. □

With this Lemma, it is in fact not too difficult to modify the proof of Theorem **??** to assure that, for a given Δ, the class of graphs with maximum degree at most Δ has a linear bound on the multipartite Ramsey number. The idea of (δ, f)-embeddable mimics

the form of the Blow-Up Lemma to achieve the same effect for other classes of graphs. More precisely:

(4.2.2) Theorem. *Let $\mathcal{G}$ be a collection of graphs that is (f, δ)-embeddable. If χ is bounded on $\mathcal{G}$ then*

$$p_m(G_1, ..., G_m) \leq A \max_i f(G_i, \frac{1}{4m})$$

for any $G_1, ..., G_m \in \mathcal{G}$ and some suitable constant A.

Proof. We let $\mathcal{G}$ be such a collection of graphs, and let ϵ be given from the choice of δ with the fact that $\mathcal{G}$ is (f, δ)-embeddable. Now choose any graphs $G_1, ..., G_m \in \mathcal{G}$ and let $N = \max_i f(G_i, \frac{1}{4km})$.

From Lemma **??**, we have, with our choice of $\epsilon' = \frac{\epsilon}{2}$, an A such that any m-coloring of $K_{AN;r_m(\chi(G_1),...,\chi(G_m))}$ contains, for some i, a monochromatic graph H that is (ϵ', δ)-super-regular on $K_{N;\chi(G_i)}$. From our choice of N, we see that G_i is embeddable into this graph H. □

We note in passing that this provides a bound for the regular Ramsey numbers well, as the chromatic number is bounded. Now from Lemma **??**, we have the following:

(4.2.3) Corollary. *Given k_0 and m, there exists a constant $A > 1$ such that*

$$p_m(G_1, ..., G_m) \leq A^{n^{k_0 - 1}}$$

for any graphs $G_1, ..., G_m$ with $\chi(G_i) \leq k_0$ and $n = \max_i |G_i|$.

Also, from Lemma **??** we have the analogue of the result of Chen and Schelp on any number of colors.

(4.2.4) Corollary. *Given p and m, there exists a positive constant A such that*

$$p_m(G) \leq A|G|$$

for any graph $G \in \mathcal{P}_p$.

4.3 Alternate Proofs

This section is motivated by a simple conjecture due to Erdős.

(4.3.1) Conjecture. $\lim_{k\to\infty}(r(k))^{1/k}$ exists.

If this limit exists, then it is known to be within $\sqrt{2}$ and 4. For bipartite Ramsey number, the analogue of this is well known. Namely:

(4.3.2) Conjecture. $\lim_{k\to\infty}(p_2(K_{n,2}))^{1/n}$ exists.

If this limit exists, then it is known to be within $\sqrt{2}$ and 2. We have yet to provide any bounds tight enough to prove this last statement, but the upper bound is accomplished in this section, and the lower bound in the next. We in fact can make a stronger conjecture.

(4.3.3) Conjecture. For a fixed k, $\lim_{k\to\infty}(p_2(K_{n,k}))^{1/n}$ exists.

The bounds given in this section and the next are strong enough to say that for a fixed k, there exists a constant A such that if this limit exists then it is between $\sqrt{2^{k-1}}$ and A.

We achieve this by presenting direct proofs, considering only the case for 2-colorings. The second proof is based on Lemma **??**, while the first is based on the following idea.

(4.3.4) Lemma. *Let integers $m > 0$ and $k > 1$ be given, and let G be any graph on $V_k(N_k(m))$, where $N_k(m) = 2^{2^{k-1}m}2^{k-2}m$. There exists set $A_1, ..., A_k$, $A_i \subseteq V_i$, such that*

each pair (A_i, A_j), $j \neq i$, *is either a complete bipartite graph or an empty graph, and* $|A_i| = m$ *for* $1 \leq i \leq k$.

Proof. We shall proceed with induction on k, beginning with the case $k = 2$.

Let G be any graph on $V_2(N_2(m))$, where $N_2(m) = m2^{2m}$. Take any set $A_2' \subseteq V_2$ with $|A_2'| = 2m$. In V_1, the pigeonhole principle guarantees that we have at least m vertices that have the same neighborhood in A_2. Let A_1 denote such a set. Every vertex in A_1 is either a neighbor of all of the vertices of A_2' or none of them. Hence at least half of the vertices of A_2' are neighbors of A_1, or half are not neighbors of all vertices of A_1. Therefore we have a set of m vertices in A_2' that form an empty graph or a complete graph with A_2. Calling this subset A_1 gives the result for $k = 2$ with the pair (A_1, A_2).

Now assume that the result is true for $k = q$ for all m with $N_q(m) = 2^{2^{q-1}m}2^{q-2}m$, and let G be a graph on $V_{q+1}(N_{q+1}(m))$ with $N_{q+1}(m) = 2^{2^q m}2^{q-1}m$. Let us note the fact that a subgraph of a complete (empty) multipartite graph is again a complete (empty) multipartite graph.

By induction and the fact that $N_{q+1}(m) = N_q(2m)$, we have sets $A_i' \subseteq V_i$, $i = 1, ..., q$, such that each pair (A_i', A_j') is either empty or complete, and each $|A_i'| = 2m$. The total number of vertices all of the A_i''s is $2qm$. In V_{q+1}, we have more than $m2^{2qm}$ vertices, and as $2^x \geq 2x$ for all integers x, there exists a set of vertices $A_{q+1} \subseteq V_{q+1}$ such that each $v \in A_{q+1}$ has the same neighborhood in each A_i' for all $1 \leq i \leq q$, and $|A_{q+1}| = m$. In the pair (A_i', A_{q+1}), we have at least half of the vertices of A_i' are either adjacent to all the vertices of A_{q+1}, or are adjacent to none of the vertices of A_{q+1}. Such a collection is then titled A_i. Then (A_i, A_{q+1}) is either empty or complete for each $i \leq q$. The result now holds with the sets $A_1, .., A_{q+1}$.

Induction then gives the result for all k and m. □

We now present the alternate proofs of the $c = 2$ case of Theorem**??**.

(4.3.5) Theorem. *Let $m, k \in \mathbb{N}$ be given. There exists an integer M such that any two coloring of the edges of $K_{M;r(k)}$, where $r(k)$ denotes the Ramsey number of k, has a monochromatic copy of $K_{m;k}$.*

Proof. (1) Given k, denote $r(k)$ by r. Let $M = 2^{2^{r-1}m}2^{r-2}m$, and χ be any 2-coloring of the edges of $K_{M;r}$ in colors, say, red and blue. By the previous lemma, in the blue subgraph we have sets $A_i \subseteq V_i$ such that $|A_i| = m$ and each pair (A_i, A_j) is either empty or complete. Consider now a copy of K_r, where each vertex v_j, ordered in any fashion, is associated a the set A_i. We two color the edges of K_r in the following way: for each edge v_iv_j, assign one color if the pair (A_i, A_j) is complete, and the other color if this pair is an empty graph. By Ramsey's Theorem, we have a monochromatic copy of K_k. Without loss of generality, assume that the associated pairs are $A_1, ..., A_k$. From the way the coloring of K_r was defined, we see that the induced subgraph from the pairs is either empty, or a complete k-partite graph. If the latter holds, then we are done; if the former holds, then in the red graph, each pair is complete and again we are done. $\square$

Proof. (2) For m and k given, and set $r = r(k)$. Pick $M = (2\binom{r}{k})^{m^{k-1}}$. Consider any 2-coloring χ of the edges of the graph $K_{M,r}$. From the definition of r, each copy of K_r has a monochromatic copy of K_k. Also, any monochromatic copy of K_k can fill this condition for at most M^{r-k} copies of K_r. There are a total of M^r copies of K_r, so that we have at least M^k monochromatic copies of K_k.

Let $V_1, ..., V_k$ be selected so that this collection has at least the average number of monochromatic copies of K_k. This average number is $M^k/\binom{r}{k}$. Also, it is true that at least half of these are in the same color, say red. Now let $G \subseteq V_1 \times ... \times V_k$ be defined by $a = (v_1, ..., v_k) \in G$ if and only if the graph on the vertices $v_1, ..., v_k$ is monochromatic

in red. As $|G| \geq M^k/(2\binom{r}{k})$, the previous Theorem **??** gives a subset of G of the form $B_1 \times ... \times B_k$ with $|B_i| = m$ for each i because

$$\frac{M^k}{(2\binom{r}{k})} = \frac{M^k}{M^{1/(m^{k-1})}} = M^{k-1/m^{k-1}}.$$

This gives a monochromatic copy of $K_{m,k}$. □

The first proof gives the bound

(4.3.6) Corollary. $p_2(K_{n;k}) \leq 2^{2^{r(k)-1}n}2^{r(k)-2}n$.

The second proof gives

(4.3.7) Corollary. $p_2(K_{n;k}) \leq (2\binom{r}{k})^{n^{k-1}}$.

The latter bound is very effective for the bipartite case, achieving the known bound of 2^m. The former achieves a simple exponential upper bound in terms of m for all (fixed) values of k. These justify the statements made in the beginning of the section about bounding any potential limits above. The lower bounds are justified in the next section.

4.4 A Lower Bound

To prove a lower bound is somewhat more complicated than in the case for the regular Ramsey numbers. However, the bipartite case is very nice, as $r(2) = 2$. We isolate this case and begin with it, noting that the methods we employ are probabilistic.

(4.4.1) Proposition. $p_m(K_{n;2}) \geq \frac{n}{e}\sqrt{m^n}$.

Proof. Let $K_{n;2}$ be given, and consider $K_{N;2}$ with $N \geq n$. If we randomly color the edges of $K_{N;2}$ with m colors, each color having probability $1/m$, then the probability that some fixed copy of $K_{n,2}$ to be monochromatic is m^{1-n^2}. There are

$$\binom{N}{n}^2$$

copies of $K_{n;2}$ in $K_{N;2}$. Treating these all independently, we see that

$$\binom{N}{n}^2 m^{1-n^2}$$

is an upper bound for the probability that some copy of $K_{n;2}$ is monochromatic. If this probability is less than one, then there exists an m-coloring of $K_{N;2}$ that admits no monochromatic copy of $K_{n;2}$. Using the inequality

$$\binom{x}{y} \leq \frac{e^y x^y}{y^y}$$

for positive integers $x \geq y$ gives the result. □

To prove the more general case, we must be able to count the number of copies $K_{n;k}$ contained in the larger graph $K_{N;r(k)}$. This turns out to be a daunting task. However, we can over count this number quite a bit and in the end still obtain a reasonable result. We present the over counting separately.

(4.4.2) Proposition. *Let $N \geq n$. Given $K_{n;k}$, there are less than*

$$\binom{Nr}{nk} \frac{(nk)!}{(n!)^k}$$

copies of $K_{n;k}$ in $K_{N;r}$, where $r \geq k$.

Proof. Let us pick any nk vertices from $K_{N;r}$. There are less than

$$\binom{nk}{n} \cdot \binom{(k-1)n}{n} \cdot \ldots \cdot \binom{n}{n}$$

copies of $K_{n;k}$ on this set of vertices. This simplifies to

$$\frac{(nk)!}{(n!)^k}$$

upon expanding each term into factorials. As there are $\binom{Nr}{nk}$ ways of choosing nk vertices in $K_{N;r}$, the result follows. □

Now the lower bound follows.

(4.4.3) Proposition. $p_m(K_{n;k}) \geq \frac{\sqrt{2m}}{er} m^{(k-1)n/2}$.

Proof. Following the same method employed in the bipartite case, we see that if

$$\binom{Nr}{nk} \frac{(nk)!}{(n!)^k} m^{1-\binom{k}{2}n^2} < 1,$$

then $N < p_m(K_{n;k})$. We have that

$$\binom{Nr}{nk} \frac{(nk)!}{(n!)^k} m^{1-\binom{k}{2}n^2} \leq \frac{(eNr)^{nk}}{(nk)^{nk}} \frac{(nk)^{nk}}{2^{(n-1)k}} m^{1-\binom{k}{2}n^2}$$

Setting the right hand side less than one gives

$$N < \frac{\sqrt{2m}}{er} m^{(k-1)n/2},$$

which proves the result. □

Bibliography

[1] N. Alon, Subdivided Graphs have Linear Ramsey Numbers, *J. Graph Theory* **18** (1994) 343-347.

[2] B. Bellobas, *Modern Graph Theory,* Springer Verlag New York, Inc. (1998).

[3] S.A. Burr and P. Erdos, On the Magnitude of Generalised Ramsey Numbers, in "Infinite and Finite Sets," Vol. 1, 214-240. Colloquia Mathematica Societatis Janos Bolyai, Vol. 10, North Holland, Amsterdam/London, 1975.

[4] V. Chvátal, V. Rodl, E. Szemerédi, and W.T. Trotter, The Ramsey Number of a Graph with Bounded Degree, *J. Combinatorial Theory, Ser. B* **34** (1983) 239-243.

[5] G. Chen and R.H. Schelp, Graphs with Linearly Bounded Ramsey Numbers, *J. Combinatorial Theory, Ser. B* **57** (1993) 138-149.

[6] D. Day, W. Goddard, M.A. Henning, and H.C. Swart, Multipartite Ramsey Numbers, *Ars. Combin.,* **58** (2001) 23-31.

[7] R.L. Graham, B.L. Rothschild, J.H. Spencer, *Ramsey Theory,* John Wiley and Sons, Inc. (1980).

[8] J.A. Hattingh, M.A. Henning, Bipartite Ramsey Theory, *Util. Math.* **53** (1998) 217-230.

[9] S. Jukna, *Extremal Combinatorics: with applications in computer science,* Springer-Verlag New York, Inc. (2001).

[10] H.A. Kierstead and W.T. Trotter, Planar Graph Colorings with an Uncooperative Partner, *J. Graph Theory* **18** (1994) 569-584.

[11] J. Komlos, G. N. Sarkozy, E. Szemerédi , Blow-Up Lemma, *Combinatorica* **17** (1997) 109-123.