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FINITE BARGAINING PROBLEMS

BY

HANJI WU

A Dissertation Submitted in Partial Fulfillment
of the Requirements for the Degree
of
Doctor of Philosophy
in the
Andrew Young School of Policy Studies
of
Georgia State University

GEORGIA STATE UNIVERSITY
2007

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ACCEPTANCE

This dissertation was prepared under the direction of the candidate's Dissertation Committee. It has been approved and accepted by all members of that committee, and it has been accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Economics in the Andrew Young School of Policy Studies of Georgia State University.

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ABSTRACT

FINITE BARGAINING PROBLEMS

BY

HANJI WU

MAY 2007

Committee Chair: Dr. Yongsheng Xu

Major Department: Economics

Bargaining is a process to decide how to divide shared resources between two or more players. And axiomatic bargaining specifies desirable and simple properties the outcome of the bargaining should satisfy and identifies the solution that produces this outcome.

This approach was first developed by John Nash in his seminal work in 1950 (Nash 1950). Since then, numerous studies have been done on bargaining problems with convex feasible set or with non-convex but comprehensive feasible set. There is, however, little work on finite bargaining problems.

In this dissertation, we study finite bargaining problems systematically by extending the standard bargaining model to the one consisting of all finite bargaining problems. For our bargaining problems, we first propose the Nash, Maximin, Leximin, Maxiproportionalmin, Lexiproportianlmin solutions, which are the counterparts of those that have been studied extensively in both convex and non-convex but comprehensive problems. We then axiomatically characterize these solutions in our context.

We next introduce two new solutions, the maximin-utilitarian solution and the utilitarian-maximin solution, each of which combines the maximin solution and utilitarian solution in different ways. The maximin-utilitarian solution selects the alternatives from the maximin solution that have the greatest sum of individuals' utilities, and the utilitarian-maximin solution selects the maximin alternatives from the utilitarian solution. These two solutions attempt to combine two important but very different ethical principles to produce compromised solutions to bargaining problems.

Finally, we discuss several variants of the egalitarian solution. The egalitarian solution in finite bargaining problems is more complicated than its counterpart in either convex or non-convex but comprehensive bargaining problems. Given its complexity in our context, we start our inquiry by investigating two-person, finite bargaining problems, and then extend some of the analysis to n-person, finite bargaining problems.

Our analysis of finite bargaining problems and axiomatic characterizations of the extensions of various standard solutions of convex/non-convex but comprehensive bargaining problems to finite bargaining problems will shed new light on the behavior of these solutions. Our new solutions will expand our understanding of the bargaining theory and distributive justice from a different perspective.

CHAPTER 1.

INTRODUCTION

Bargaining Problems

In the problem of distributing a fixed amount of goods, how should we determine who gets what? A bargaining is a process to decide how to divide a shared resource between two or more participants.

A pure bargaining problem is described as follows: a group of two or more participants is faced with a set of feasible alternatives, and any alternative can be the outcome of the problem if a unanimous agreement among all the participants is reached at that alternative. In the event of no unanimous agreement, a given disagreement alternative is specified as the outcome. If at least two of the participants differ over which alternative is most preferable, there is a need for bargaining and negotiation over which alternative should be agreed upon.

To summarize, there are four components in a bargaining problem.

1. Players: a group of two or more participants who distribute a fixed amount of goods. Each player has the ability to veto any outcome different from the disagreement outcome since unanimity is required for reaching an agreement.

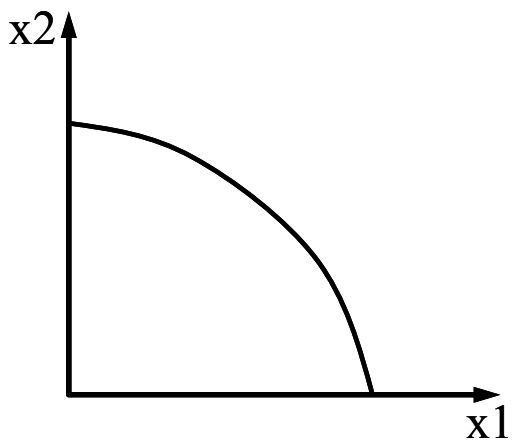
2. Feasible set: The set of all feasible alternatives, any one of which can be the

outcome of the problem if it is specified by the unanimous agreement. The disagreement point defined below is one element in the feasible set. An alternative in the feasible set of a bargaining problem can be represented by an n-tuple of real numbers, where the i-th component is the amount of the goods the player i can get from that outcome. Another interpretation of an alternative in the feasible set is that the i-th component is the utility of the player i for the alternative in question. In our model, we take the latter interpretation. A feasible set in our model is represented by a subset of the n dimensional Euclidean space.

According to different economic settings, the feasible set can be one of the following:

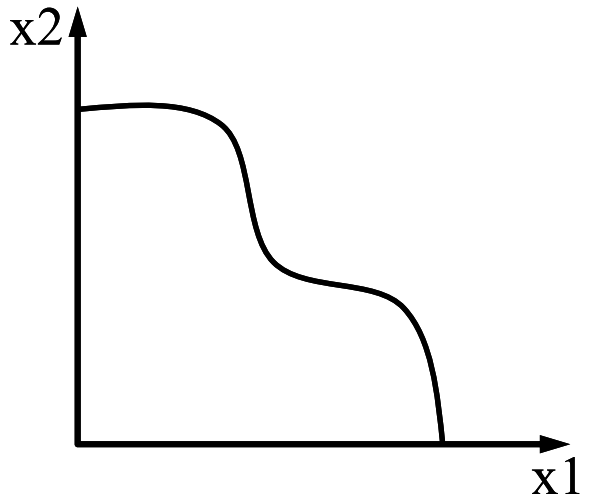
a. Convex feasible set: Under the assumption that the players may agree to randomize between alternatives if they so choose and the utility is disposable, the feasible set S is typically convex as illustrated in Figure1. We can see in Figure 1, any convex combination of any two outcomes in the set lies in the set.

Figure 1. Convex Feasible Set.



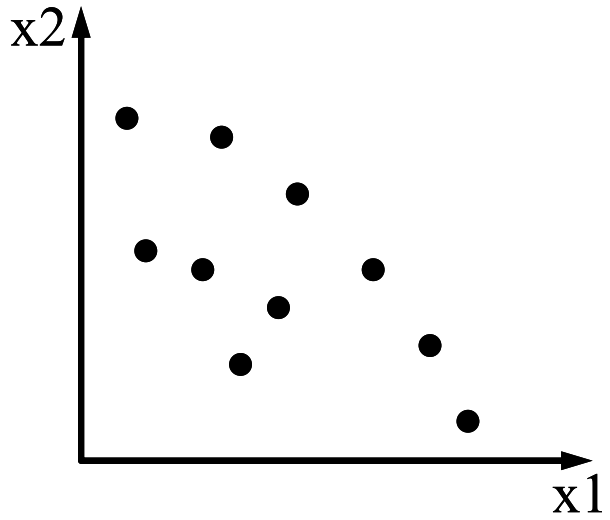
b. Non-convex but comprehensive feasible set and finite feasible set: If we drop the assumption that the players can randomize between outcomes, the feasible set faced by the players will not always be convex. Figure 2 illustrates a feasible set which is non-convex but comprehensive.

Figure 2. Non-Convex but Comprehensive Feasible Set.



c. If we further relax the assumption that the utility is not disposable, the players will face a more rigid feasible set in which the outcomes in the feasible may be discrete and finite. Figure 3 illustrates a feasible set which is discrete and finite.

Figure 3. Finite Feasible Set



3. Solution: A solution to a bargaining problem specifies the alternatives that the players unanimously agree on. It is the result of the bargaining and final decision of who getting what.

4. Disagreement point: In the event that no unanimous agreement is reached, a given disagreement outcome is specified as the final outcome. That means that any individual has the option of choosing the disagreement outcome and ensure that it will be the outcome of the game by vetoing other alternatives. A player would always compare the payoff from an alternative with that from the disagreement point and decide to agree on which alternative to be the final outcome.

Axiomatic Approach

There are two approaches to study bargaining problems and both are presented by Nash: axiomatic and strategic. The axiomatic approach specifies some desirable properties that the outcome of a bargaining process should satisfy and on the other hand identifies the bargaining rule that produces the outcome.

Concentrating on two-person bargaining problems, John Nash (1950) took an axiomatic approach and presented a framework in which a unique feasible alternative is selected as the "solution" to a given bargaining problem. Thus, the solution to a problem is a rule which selects the outcome from the problem. Nash (1950) proposed that a solution possesses the following properties:

1. Efficiency: Efficiency requires that the solution select an outcome which is

efficient. This property restricts the solution to the points on the "contract curve" as described in Edgeworth (1881) and Von Neumann-Morgenstern (1944).

2. Symmetry: If a bargaining problem is symmetric in respect to disagreement point, that is, the problem is in disagreement point symmetry, each player should get the same level of utility because the bargaining power of each player is the same.

3. Scale invariance: The solution is invariant with respect to a linear transformation of the utility. In Nash's solution, utility function satisfies expected utility theory and is unique up to a linear transformation.

4. Contraction independence, also called independence of irrelevant alternatives: It says that the outcome of a bargaining problem will continue to be selected when the bargaining problems shrinks and the initially selected outcome is still available.

Nash (1950) proved that the solution satisfying the above four properties is uniquely equivalent to the outcome that maximizes the product of individual utilities of the participants. This solution has become known as the Nash solution.

Another reason for an axiomatic approach is that an axiomization of a solution can help us to understand the solution by breaking it down into several pieces, which are the corresponding properties uniquely determining the solution. Those properties are typically simple and easily understood in comparison with the solution itself.

The properties can also help us to test the validity of a solution. In reality, it is hard to measure the utility of every player. So we can hardly tell if the result of a bargaining is the maximization of the product of all players' utility or not. But the properties that uniquely determine the Nash's solution are more easily to be tested

conceptually as well as empirically. If we can observe any of the properties failing the empirical test, we can then conclude that the solution which possesses this property can not pass the empirical test.

Existing Studies

Convex Problems

Since Nash (1950)'s seminal work, many economists have contributed to bargaining theory by challenging the four properties that uniquely determine Nash's solution, or by putting forward other reasonable properties that Nash's solution fails to satisfy.

In the four axioms which uniquely determine Nash's solution, the scale invariance and contraction independence have been the focus of criticism. These two axioms have been challenged on conceptual grounds as well as by empirical tests, for example, see Crott (1971), De Menil (1971), Fouraker and Siegel (1963), Nydegger and Owen (1974), O'Neill (1976), Rapoport, Guyer and Gordon (1976), Rapoport, Frenkel and Perner (1977), Siegel and Fouraker (1960), and Stone (1958). There are various other solutions suggested and studied, and alternative properties are considered.

In response to the property of scale independence, Shapley (1969) argued that the solution should be insensitive to a more general utility transformation which preserves the ordinality of utility functions of the players. That is, the bargaining solution should be invariant of any increasing transformation. But his studies showed that the independence of ordinal transformations is a very strong requirement and it is

in contradiction with some other standard properties, such as strong individual rationality and contraction independence. Roth (1979c) and Roth and Malouf (1979) presented a bargaining solution in a two-player model which possesses the independence will still be satisfied if the transformation (1) preserves each player's ordinal preference and (2) preserves information about which player makes larger gains at any given point. This independence property as well as other standard property specifies an outcome that is the strongly Pareto optimal point and that maximizes the minimum gains available to the players. This solution is the counterpart of the leximin solution in our finite bargaining problems to be introduced in Chapter IV. We will characterize this solution with different properties in Chapter 4.

Nash's contraction independence models a bargaining process which can proceed by first narrowing down the original set of feasible alternatives to some smaller set which includes the outcome identified by the solution, without changing the outcome. This axiom came under a lot of criticism (Luce and Raiffa 1957，128). In particular, Kalai and Smorodinsky (1975) and Kalai (1977b) criticized this axiom by pointing out that the narrowing of the original set may cost the bargaining power of a player in the bargaining problem and questioned the validity of this axiom.

Figure 4, Figure 5 and Figure 6 illustrate two bargaining problems with the same disagreement point (0,0).

Figure 4 illustrates a problem with the feasible set: S_1=convex hull $\{(0,1),(1,0),(0,0)\}$.

Figure 4. A Bargaining Problem.

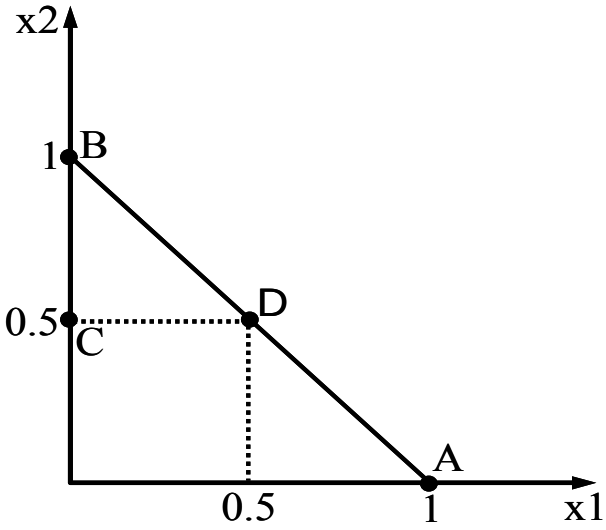


Figure 5 and Figure 6 both illustrate a narrowed problem compared with the problem in Figure 4. The feasible set of the problem is S_2 =convex hull $\{(0,1),(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},0),(0,0)\}$.

Figure 5. Narrowed Bargaining Problem and Nash Solution

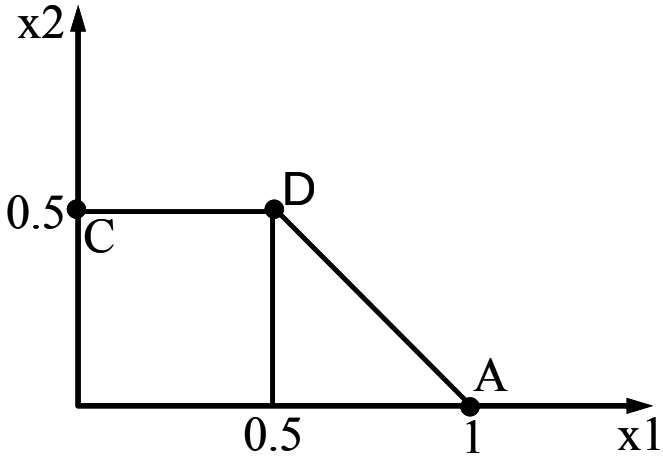
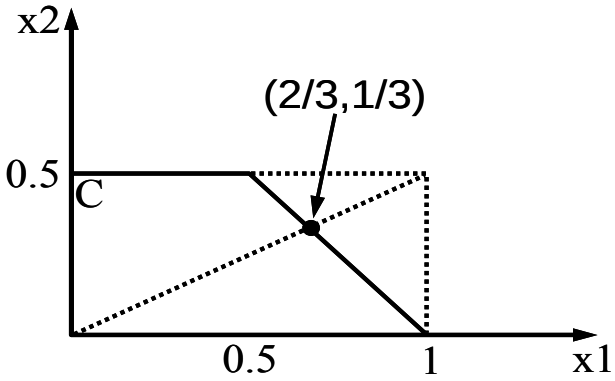


Figure 6. Narrowed Bargaining Problem with Kalai-Smorodinsky Solution



Nash's solution of $(0, S_1)$ is: $\left(\frac{1}{2}, \frac{1}{2}\right)$. By axiom of contraction independence, the solution of $(0, S_2)$ is $\left(\frac{1}{2}, \frac{1}{2}\right)$, too, as in Figure 5. The alternatives in the set BCD have no impact on the result. But in problem $(0, S_2)$, player 2 has lost his favorable alternatives and player 1 has a good reason to demand that he or she should get more. In order to overcome this difficulty, Kalai-Smorodinsky presented a solution which depends not only on the disagreement payoff but also on the maximal payoffs each player might hope to receive. Thus Kalai-Smorodinsky solution does not possess the property of the insensitivity of so called irrelevant alternative available to participants.

From the definition of Nash solution and Kalai-Smorodinsky solution, for the problem S_1, Nash solution and Kalai-Smorodinsky solution are the same:

$$F^N(S_1) = F^{KS}(S_1) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

For problem S_2, Nash solution satisfies the axiom of contraction independence,

so $F(S_2) = F(S_1) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the loss of favorable alternatives in area BCD doesn't change the bargaining power of Player 2.

Figure 6 illustrates $F^{KS}(S_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$. Kalai-Smorodinsky solution is monotone to the maximal payoffs each player might receive.

Besides the solutions mentioned above, philosophers and other economists have also proposed a variety of bargaining solutions for "just" allocations. For example, egalitarian solution chooses an outcome giving equal utility to each player and lying on the utility frontier. The egalitarian solution satisfies these properties: weak efficiency, symmetry, and contraction independence and fails scale invariance. As another example, the utilitarian solution chooses an outcome maximizing the sum of all players' utilities. The utilitarian solution satisfies these properties: efficiency, contraction independence.

Non-convex Bargaining Problems

The existing studies on bargaining problems mostly concentrate on those with convex feasible set obtained by the assumptions that each player agrees to randomize between outcomes and that the utility is disposable. The former assumption is also required in the expected utility theory, which plays an important role in Nash's solution. The convex feasible set has its limitations in several contexts in economics. Non-convex bargaining problems can arise in many economic contexts, for example, when individuals are non-expected utility maximizers. They can also arise naturally in bargaining problems when individuals are not characterized by their utilities but by

their functionings and capability sets presented in the sense of Sen (1985). Also see Xu and Yoshihara (2005) for such cases.

In recent years, non-convex bargaining problems have been attracted much attention by researchers. See for example, Kaneko (1980), Herrero (1989) , Conley and Wilkie (1991; 1996), Mariotti (1998), Xu and Yoshihara (2006), and Zhou (1997). These studies defined and characterized the Nash solution, the Kalai-Smorodinsky solution and the egalitarian solutions for comprehensive and compact but not necessarily convex bargaining problems. These studies on non-convex bargaining problems have extended our insights into the solutions of the bargaining problems. On the other hand their focus is still on comprehensive and compact problems, the definitions and characterizations of solutions are fairly similar to those for bargaining problems with convex feasible sets. Meanwhile, studies on non-convex and finite bargaining problems are still few in the literature.

A Few Studies on Finite Bargaining Problems

There are some discussions on finite bargaining problems in the literature. For example, Mariotti (1998) characterized the Nash solution for domains including or consisting of feasible sets with finite number of alternatives. Following a similar approach as Mariotti (1998), Tanaka and Nagahisa (2002) characterize the Kalai-Smorodinsky solution (Kalai and Smorodinsky 1975) for the finite domain. There is, however, a lack of systematic study on finite bargaining problems. This dissertation provides a thorough and systematic study of finite bargaining problems.

Motivation of Our Studies on Finite Bargaining Problems

The Economic Settings of Finite Problems

Our interests in finite bargaining problems arise from two observations.

First, in most practical bargaining situations, the alternatives available to players are often finite, and the players can not always randomize the alternatives available. Thus the assumption of convex feasible set is not attainable. For example, it is often the case that, when a union bargains with a firm, there are only a few options available on the table for both parties to consider.

Secondly, finite bargaining problems can also arise with indivisible goods and no side payments. There are extensive studies on indivisible goods in the literature, see, for example, Alkan, Demange and Gale (1991), Maskin (1987), Quinzii (1984), Svensson (1987) . These studies mainly focus on allocation rules. In this essay, we study similar issues from the perspective of bargaining theory. In an economy with indivisible goods, extensions of conventional bargaining theory to non-convex but comprehensive problems are therefore unsatisfactory because, when goods are indivisible, it is not plausible to make the assumption that we can obtain any level of utility in our imagination since the goods with which we generate utility are discrete. As a consequence, finite bargaining problems arise naturally in such settings.

An Example

We can find the example of finite bargaining problems in air traffic control. In air traffic control, every flight is assigned departure and arrival slots, flights routes and gates. Air traffic controllers use strict rules of thumb. For example, Ball et al. (2000) evaluated two rules: (1) first come first serve used often for aircraft that want

to depart or have missed their slot: They are directly put at the end of the queue of aircraft waiting to be scheduled; And (2) ground delay programs: if an airport is at its maximum capacity, all incoming aircraft are delayed an equal amount of time. These rules are strictly executed by air traffic controller and they are hardly to be persuaded to deviate from these rules, though sometimes deviation of the rules results in more efficiency. That's because such a deviation requires the judge of the air traffic controller and this kind of judge may be inconsistent in different circumstance, and would violate an important underlying principle in air traffic control: fairness. These rules of thumb provide some assurance that different airlines will be treated equally.

Jonker, Meyer and Dignum (2005) described one condition under which the air traffic controller would be willing to deviate from their rules of thumb: when the multi-player bargaining could provide a solution. The airline companies would negotiate among each other about plan changes. When doing this, they could explicitly compare each other utilities and adapt their behavior accordingly, and could achieve higher efficiency. This is a finite bargaining. The resources they divide are departure and arrival slots, flights routes and gates. And during their bargaining, there are no side payments in them.

Comparison of Existing Studies and Finite Bargaining Problems

To study finite bargaining problems, we first note that the definition of a solution may be different from the classical bargaining problems. This is because a solution obtainable in the classical problems may not be feasible for discrete and finite problems. For example, the Kalai-Smorodinsky solution and the egalitarian

solution defined in convex problems may not always exist in finite problems. Secondly, in finite problems, a solution is typically multi-valued. Thirdly, some properties, for example, the scale invariance axiom, characterizing solutions for convex bargaining problems may not be applicable to finite problems and may call for different interpretations even when they continue to be applicable in finite problems.

From the analysis above we can see, there is a necessity of our study on finite bargaining problems.

The Structure of This Essay

In this essay, we will provide a systematic study of finite bargaining problems by first formalizing the problem, then defining various solutions to bargaining problems in a discrete and finite domain, exploring various properties to be imposed on a solution to finite bargaining problems, and characterizing alternative solutions proposed in our context. The specific solutions that we are interested in are the Nash, maxmin, leximin, maxiproportionalmin, lexiproportionalmin, utilitarian, maximin-utilitarian, and utilitarian-maximin solutions, and several variants of the egalitarian solution.

The essay is organized as follows: In Chapter II, we model the finite problem and provide notations and some definitions used in this essay. In Chapter III, we present the properties which we use in the characterizations of various solutions. Chapter IV defines and characterizes our first set of solutions for finite bargaining problems. In Chapter V, we study the utilitarian, maximin-utilitarian and utilitarian-maximin solutions. Discussions of several variants of the egalitarian

sol,ution are given in Chapter V. Chapter VI summarizes the dissertation by presenting a brief conclusion and outlining our main contribution to the understanding of axiomatic bargaining problems.

CHAPTER 2.

MODEL, NOTATIONS AND SOME DEFINITIONS

The Mathematic Model of Finite Problems

Let $N = \{1, \ldots, n\}$ be the set of players with $n \geq 2$. Let R_+ be the set of all non-negative real numbers, R_+^n be the $n-$dimensional non-negative Euclidean space.

A bargaining game is described by a set of players N and a pair (A, d), where $A \subset R_+^n$ is the feasible set with each element in A being a utility vector specifying payoffs to each player and $d \in A$ is the disagreement point. We normalize d to 0. Let $\mathcal{B} = \left\{A \subseteq R_+^n \middle| 0 \in A, A \cap R_{++}^n \neq \phi \text{ and } 2 \leq \# A < \infty\right\}$. Each A in $\mathcal{B}$ is considered as a (normalized) bargaining problem with $0 \in R_+^n$ as the disagreement point, and $\mathcal{B}$ is the family of all finite normalized bargaining problems each of which contains at least one element such that every player's utility is strictly greater than the disagreement point $d = 0$. That is, for each $A \in \mathcal{B}$, there exists $a \in A$ such that $a_i > 0$ for all $i \in N$.

A solution to a bargaining problem is a function F defined over $\mathcal{B}$ such that $F(A)$ is a non-empty subset of A for any $A \in \mathcal{B}$. That is, a solution is a rule which assigns to each bargaining problem a set of utility feasible vectors of the problem.

Some Notations and Definitions

Let π be a permutation of N. The set of all permutations of N is denoted

by Π..

As usual, for every $x, y \in R_+^n$, we write $x > y$ to mean $[x_i \geq y_i$ for all $i \in N$ and $x \neq y]$, and $x >> y$ to mean $[x_i > y_i$ for all $i \in N]$.

For each $x \in R_+^n$, define $\Theta(x)$ as the vector $(x_{\theta(1)}, \ldots, x_{\theta(i)}, \ldots, x_{\theta(n)})$, where $\Theta \in \Pi$, such that $(x_{\theta(1)} \leq \ldots \leq x_{\theta(i)} \leq \ldots \leq x_{\theta(n)})$. For each $x \in R_+^n$, we sometimes write $\Theta(x)$ as $(\Theta(x)_1, \Theta(x)_2, \ldots, \Theta(x)_n)$, with $\Theta(x)_1 \leq \Theta(x)_2 \leq, \cdots, \leq \Theta(x)_n$.

Definition 2.1: $x \geq_{lexi\min} y$**.** For every $x, y \in R_+^n$, $x \geq_{lexi\min} y \Leftrightarrow [(\Theta(x)_1 > \Theta(y)_2)$, or $(\Theta(x)_1 = \Theta(y)_1, \Theta(x)_2 > \Theta(y)_2)$ or… or $(\Theta(x)_1 = \Theta(y)_1, \ldots, \Theta(x)_{n-1} = \Theta(y)_{n-1}, \Theta(x)_n > \Theta(y)_n]$.

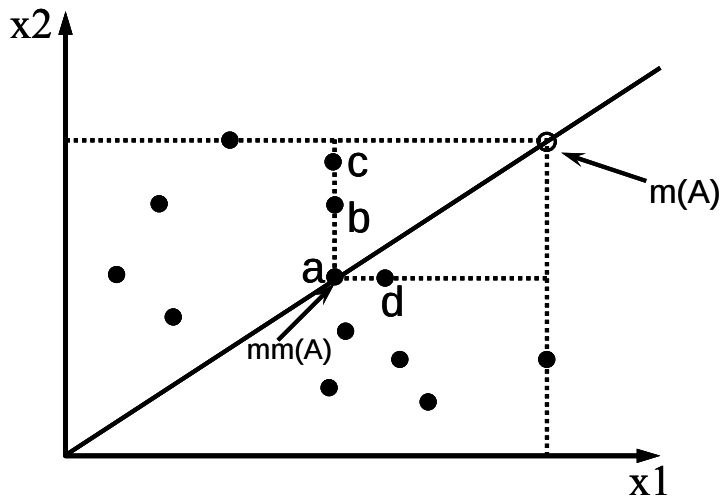
Definition 2.2. $k-$**symmetric.** For each $k \in \{1, \ldots, n-1\}$ and each $x \in R_+^n$, we say that x is $k-$symmetric if $\Theta(x)_k = \Theta(x)_{k+1} = \ldots = \Theta(x)_n$. Note that, for $k = 1$, x is $k-$symmetric if $x_1 = x_2 = \ldots = x_n$.

Definition 2.3. Ideal point $m(A)$**:** For each bargaining problem $A \in \mathcal{B}$, let $m_i(A) \equiv \max\{a_i \mid (a_1, \ldots, a_i, \ldots a_n) \in A\}$, and $m(A) \equiv (m_1(A), \ldots, m_n(A))$. The point $m(A)$ is commonly called the ideal point of A. Clearly, $m(A) >> 0$ for each $A \in \mathcal{B}$.

Definition 2.4. **Center poin**t: We define $mm: \mathcal{B} \rightarrow R_+^n$, such that $mm(A) = \alpha$, and $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \max_{x \in A} \min_{i \in N} \{(x_i)$. We call the point $mm(A)$ the center point of the problem.

The ideal point and center point of a problem are illustrated in Figure 7.

Figure 7. Ideal Point and Center Point



Definition 2.5. Symmetric problem: For each $A \in \mathrm{B}$, a bargaining problem is said to be symmetric if, for every $a \in R^n_+$, $a \in A \Rightarrow \pi(a) \in A$ for all $\pi \in \Pi$.

Definition 2.6: Pigou-Dalton domination: For every $x, y \in R^n_+$, we say that x Pigou-Dalton dominates y if for some $\varepsilon > 0$ and some $i, j \in N$, $[y_i + \varepsilon = x_i \leq x_j = y_j - \varepsilon$ and $y_k = x_k$ for all $k \in N \setminus \{i, j\}]$.

Comparing two utility vectors, x, y, we can see that the sum of the player's utilities in alternative x is the same as that in alternative y. When a small amount of utility is transferred from a better-off player to a worse-off player while keeping their relative ranking in terms of utilities unchanged, we may argue that the inequality measured in terms of utility in y is less than that in x. In other words, y is more equitable than x in the utility space. The Pigou-Dalton domination in our context is in the same spirit as the Pigou-Dalton transfer in the discussion of income inequality. See, for example, Sen (1985, 27).

CHAPTER 3.

PROPERTIES

In this chapter we present the properties that are used to characterize the various solutions to our finite bargaining problems.

Efficiency Properties

First two are standard efficiency properties.

Axiom 3.1. Efficiency (E): For any $A \in \mathcal{B}$ and any $a \in F(A)$, there is no $x \in A$ such that $x > a$.

Efficiency is the standard axiom which requires that the solution satisfies the Pareto efficiency. That is, for any bargaining problem, there is no outcome in the feasible set, such that, a movement from the solution to that outcome can make at least one player better off, without making any other players worse off.

Axiom 3.2. Weak Efficiency (WE): For any $A \in \mathcal{B}$ and any $a \in F(A)$, there is no $x \in A$ such that $x >> a$.

Weak efficiency is another standard efficiency type axiom and is weaker than efficiency. It requires that the solution is the one where there is no outcome in the feasible set that would be strictly preferred by all agents.

Some Standard Properties

The following properties are extensions of several standard properties first described in convex bargaining problems.

Axioim 3.3. Scale Invariance (SC): For all $A \in \mathcal{B}$ and all $\alpha \in R^n_{++}$, if $\alpha A = \{(\alpha_i a_i)_{i \in N} : a \in A\}$ then $F(\alpha A) = \{(\alpha_i a_i) : a \in F(A)\}$.

This axiom is an extension of its counterpart for Nash's (1950) original axiom of scale invariance. In our context, it maintains the claim that measuring utility is invariant to the cardinal scale. Note that each player can use a different scale to measure utility implying that there is no interpersonal comparison of players' utilities and the utility is cardinal.

Axiom 3.4. Symmetry (S): For any $A \in \mathcal{B}$, if A is symmetric, then $[a \in F(A) \Rightarrow \pi(A) \in F(A)$ for all $\pi \in \Pi]$.

In Nash's (1950) original model, Symmetry property requires that, in a symmetric problem, each player gets exactly the same level of utility because each player has the same bargaining power. In our finite bargaining problem, even if the problem is symmetric, the outcome in which each player gets the same level of utility may not be available. We can not distinguish one player from another in a symmetric problem, a solution should treat players symmetrically in a symmetric problem. This version of the conventional Symmetry property is sometimes called Anonymity as well.

Axiom 3.5 Domination (D): For any $A \in \mathcal{B}$ and any $x \in A$, if $x > a$ for some $a \in F(A)$, then $x \in F(A)$.

Domination says that, for any bargaining problem A, if there is an alternative x in A such that x is greater than some outcome a in the solution $F(A)$, then x must be a part of the solution of A as well.

Axiom 3.6. Strong Individual Rationality (SIR): For each $A \in \mathcal{B}$, if $a \in F(A)$ then $a >> 0$.

Strong Individual Rationality requires that a solution to any bargaining problem must assign positive utility for every player. This is a slightly stronger requirement than the conventional axiom of individual rationality which only requires that a solution assign non-negative utility to every player. In our context, given that each problem contains at least one alternative given positive utility for each player, this is a fairly reasonable requirement.

Contraction Independence

In this subsection we introduce two properties that relate to the independence of irrelevant alternatives originally proposed by Nash (1950).

Axiom 3.7. Contraction Independence (CI): For any $A, B \in \mathcal{B}$, if $B \subseteq A$ and $B \cap F(A) \neq \phi$, then $F(B) = B \cap F(A)$.

This axiom is an extension of the independence of irrelevant alternatives which first appeared in Nash (1950). Its intuition is that an alternative which is not a part of the solution is not a relevant solution candidate when the problem "shrinks". If we remove alternatives that we did not choose as a part of the solution to a bargaining problem, we might desire that our solution to the new bargaining problem remains the same as that to the original problem. Precisely, it requires that, if the intersection of the problem after the contraction and the solution of the original problem is not empty, then this intersection is the solution to the problem after contraction.

Axiom 3.8. Weak Contraction Independence (WCI): For any $A, B \in \mathcal{B}$ with

$m(A) = m(B)$, if $B \subseteq A$ and $B \cap F(A) \neq \phi$, then $F(B) = B \cap F(A)$.

(WCI) has been used for both convex (Yu 1973) and non-convex (Xu and Yoshihara 2006) problems. Compared with ordinary contraction independence, it is weakened by additional requirement that the ideal points of the original problem and the problem after the contraction are the same.

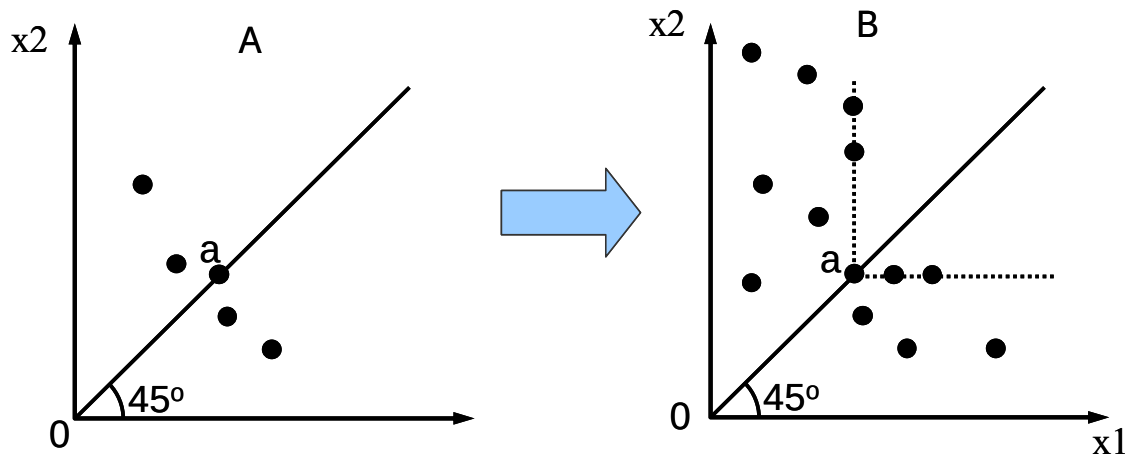
Expansion Properties

The expansion properties discussed below deal with the consistency of a solution when we enlarge the problem by adding new alternatives to the original problem.

Axiom 3.9. Expansion Consistency (EC): For each $A, B \in \mathcal{B}$ with $A \subseteq B$, if A is symmetric, $\{a\} = F(A)$ and a is weakly efficient on B, then $a \in F(B)$.

Expansion Consistency requires that, given a symmetric problem A that has a as the unique solution, when A is enlarged to B so that B is a superset of A, and if a continues to be weakly efficient on B, then a should be in the solution of B.

Figure 8. Expansion Properties.



The intuition behind expansion properties, take expansion consistency for example, is that, when the outcome of a problem which we denote with a, has some "desirable quality"-being the unique solution to a symmetric problem, then if we enlarge the original problem to a larger one by adding other alternatives, and if, in the new problem, there is no alternative that dominates a, we can not exclude a from the solution to the new problem.

The spirit of the expansion consistency is similar to a property considered by Thomson and Myerson (1980) discussed below:

Independence of Undominated Alternatives (Thomson and Myerson 1980): For each $A, B \in \mathcal{B}$, for every $a \in F(A)$, if $A \subseteq B$ and a is weakly efficient on B, then $a \in F(B)$.

Several variants of expansion consistency are introduced and discussed below.

Axiom 3.10. Expansion Independence (EI): For each $A, B \in \mathcal{B}$ with $A \subseteq B$, and for some $k \in \{1,2,\ldots,n-1\}$, if $\{a\} = F(A)$, a is $k-$symmetric, [for each $b \in B \setminus A$, $\Theta(b)_1 \leq \Theta(a)_1, \ldots, \Theta(b)_k \leq \Theta(a)_k$], and a is efficient on B, then $a \in F(B)$.

Expansion Independence is in the same spirit as Expansion consistency, and requires that, when a given problem A has a unique solution a which is $k-$symmetric for some k, if A is enlarged to B in a certain way so that a is efficient on B and the added alternatives are not lexicographically greater than a, then a should be in the solution of B.

Axiom 3.11. Restricted Expansion Consistency (REC): For each $A, B \in \mathcal{B}$

with $A \subseteq B$, if A and B are both symmetric, $\{a\} = F(A)$ and a is weakly efficient on B, then $a \in F(B)$.

Restricted Expansion Consistency is a restricted version of (EC) in that, it restricts the applicability of the axiom to the situation in which the original problem and the enlarged problem are both symmetric.

Axiom 3.12. Weak Expansion Independence (WEI): For each $A, B \in \mathcal{B}$ with $A \subseteq B$ and $m(A) = m(B)$, and for each $k \in \{1,2,\ldots,n-1\}$, if $\{a\} = F(A)$, a is $k-$symmetric, [for each $b \in B \setminus A$, $\Theta(b)_1 \leq \Theta(a)_1, \ldots, \Theta(b)_k \leq \Theta(a)_k$], and a is efficient on B, then $a \in F(B)$.

(WEI) is weaker than (EI) by restricting the applicability to the bargaining problems A and B with $m(A) = m(B)$.

Axiom 3.13. Efficient Expansion Consistency (EEC): For each $A, B \in \mathcal{B}$ with $A \subseteq B$, if A is symmetric, $\{a\} = F(A)$ and a is efficient on B, then $a = F(B)$.

Efficient Expansion Consistency requires that, given a symmetric problem A that has a as the unique solution, if A is enlarged to B so that B is a superset of A and if a continues to be efficient on B, then a should still be the unique solution of B.

Pigou-Dalton Monotonicity

Axiom 3.14. Pigou-Dalton Monotonicity (PDM): For each $A \in \mathcal{B}$ and $a \in A$, and every $x \in R^n_+$ with $x \notin A$, if x Pigou-Dalton dominates a, then

(i) $a \in F(A) \Rightarrow x \in F(A \cup \{x\})$.

(ii) $a \in F(A)$ while $\{a\} \neq F(A) \Rightarrow F(A) \cap F(A \cup \{x\}) \neq \phi$.

Axiom (PDM) requires that, if an alternative is Pigou-Dalton dominates an outcome of a bargaining problem, that alternative will be in the solution to the problem which is the combination of the original problem with that alternative. And if there are more than one outcomes in the solution to the original problem, the combined problem can not exclude all the outcomes from the original problem. The new problem will keep at least one outcome from the original problem.

Axiom 3.15. Weak Pigou-Dalton Monotonicity I (WPDMI): For each $A \in \mathcal{B}$ and $a \in A$, and every $x \in R_+^n$ with $x \notin A$, if x Pigou-Dalton dominates a, then

(i) $\{a\} \in F(A) \Rightarrow x \in F(A \cup \{x\})$.

(ii) if $mm(A) = mm(A \cup \{x\})$,

$[\{a\} \in F(A) \text{ but } \{a\} \neq F(A)] \Rightarrow [F(A) \cap F(A \cup \{x\}) \neq \phi]$.

The first part of (WPDMI) is same as axiom (PDM) and requires that, if an alternative is Pigou-Dalton dominates a solution to a problem, that alternative will be a solution to the problem which is the combination of the original problem with that alternative. But the second part of monotonicity will be kept when the center point doesn't change after the inclusion of the Pigou-Dalton dominant point. If the center point changes after the inclusion, that means the minimum utility increases in the Pigou-Dalton dominant alternative, so the Pigou-Dalton dominant alternative is superior to all the solutions of the original problem and maybe the unique solution to the new problem.

Axiom 3.16. Weak Pigou-Dalton Monotonicity II (WPDMII): For each $A \in \mathcal{B}$ and $a \in A$, and every $x \in R_+^n$ with $x \notin A$, if x Pigou-Dalton dominates

a, then $[\{a\} \in F(A) \Rightarrow x \in F(A \cup \{x\})]$, and if $mm(\{0,a\}) = mm(\{0,a,x)$,

$[\{a\} \in F(A)$ but $\{a\} \neq F(A)] \Rightarrow [F(A) \cap F(A \cup \{x\}) \neq \phi]$.

(WPDMII) is similar to (WPDMI), and the first part is the same as (PDM), but the second part of monotonicity requires the minimum of the utility of Pigou-Dalton dominant point doesn't exceed that of the solution to the original problem, thus the added Pigou-Dalton dominant alternative is not superior sufficiently to all the solutions of the original problem, and can not exclude all of them from the solution to the new problem.

Other Properties

Axiom 3.17. Exclusion (EX): For every $A, B \in \mathcal{B}$, if $mm(A) \neq mm(B)$, then $F(A) \cap F(B) = \phi$.

Exclusion requires that if the center points of two different problems are different, there is no common element in their solution.

The center point of a problem contains of information about the problem. As illustrated in Figure 3-2, a low level of center point indicates that the problem either has very unequal distributions of utility (Figure 9) or the utility level is very low (Figure 10). If the center points of two problems are different, these two problems are considered to be too different to have common element in their solutions.

Figure 9. Center Point and Inequality

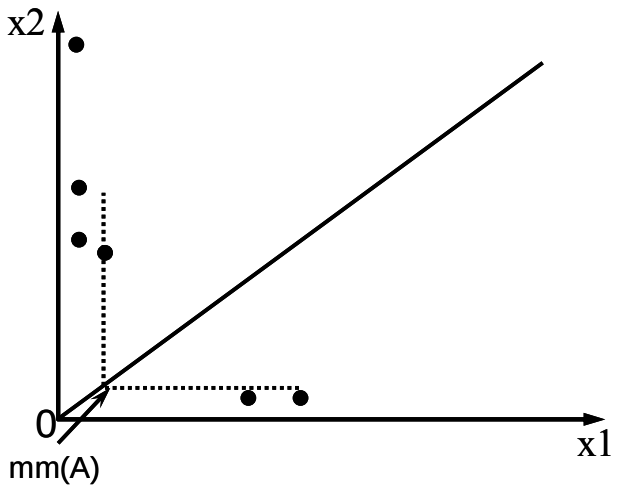
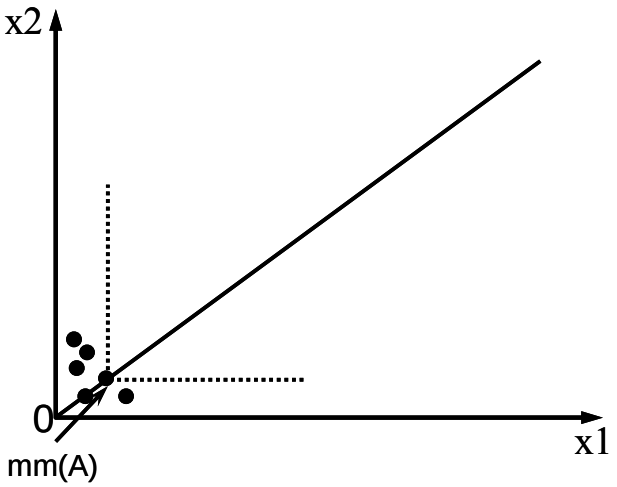


Figure 10. Center Point and Low Level of Utilities



Axiom 3.18. Replacement Independence (RI): For each $A \in \mathcal{B}$ and $b \in A$, and every $x \in R^n_{++}$ with $x \notin A$, if x Pigou-Dalton dominates b for some b in $A \backslash F(A)$, if $mm(\{0,a\}) = mm(\{0,a,x\})$, then $a \in F(A) \Rightarrow a \in F((A \setminus b) \cup \{x\})$.

Replacement Independence requires that an alternative will still belong to the solution after a Pigou-Dalton Dominant point of any other alternative in the original problem replaces that alternative and such a replace does not go too far that the minimum utility that a player can get will not exceed the minimum utility in the

solution to the original problem. We consider such an alternative and the Pigou-Dalton domination of that alternative are similar to each other and the replacement does not change the outcome.

Axiom 3.19. Hammond Equity (HE): For each $A \in \mathcal{B}$ and every $x, y \in R_+^n$, with $x \in A$ and $y \notin A$, if $x_i \leq y_i \leq y_j \leq x_j$ with $x_k = y_k$ for all $k \neq i, j$, then $x \in F(A) \Rightarrow y = F(A \cup \{y\})$.

Hammond Equity was first introduced by Hammond (1976) in a social choice context. According to Hammond Equity, one alternative is considered to be more equal than another if the difference between two players becomes smaller while all the others' are the same. An alternative is dominated by another alternative which is more equal.

CHAPTER 4.

EXTENSIONS OF CLASSIC SOLUTIONS

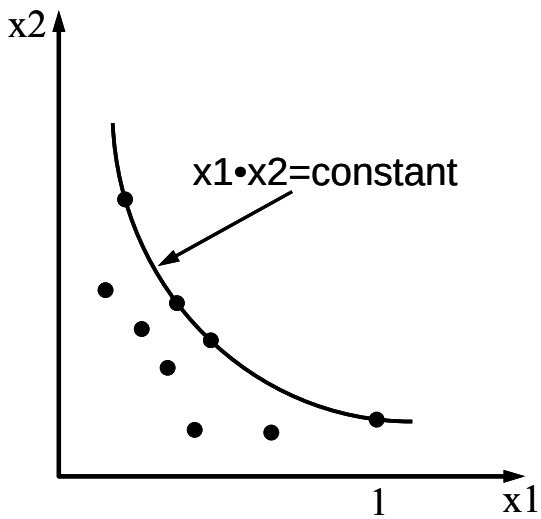
In this chapter we first define the various solutions which have been studied in convex and non-convex but comprehensive problems. Specifically, we extend the Nash solution, the egalitarian solution and the Kalai-Smorodinsky solution to out finite problems. Then, we characterize these solutions respectively with the appropriate properties discussed in chapter III.

Nash Solution

Definition of Nash Solution

Our definition of the Nash solution is the same as that in convex problems: alternatives that maximize the product of player's utilities. Figure 11 illustrates the Nash solution in a 2-player bargaining problem.

Figure 11. Nash Solution



The following definition is straightforward extensions of the Nash solution in standard convex and non-convex bargaining problems.

Definition 4.1: Nash solution

A bargaining solution F over $\mathcal{B}$ is the Nash solution if, for all $A \in \mathcal{B}$, $F(A) = \{a \in A \mid \Pi_{i \in N} a_i \geq \Pi_{i \in N} x_i \text{ for all } x \in A\}$.

The above Nash solution is a standard extension of the Nash solution for both convex and non-convex but comprehensive problems. Our notion of the Nash solution is similar to the one proposed by Kaneko (1980), Mariotti (1998), and Xu and Yoshihara (2006) for non-convex but comprehensive bargaining problems, and is identical to the Nash solution considered by Mariotti (1998) for finite bargaining problems. Other proposals of the Nash solution for non-convex problems include Conley and Wilkie (1991; 1996) and Zhou (1997), whose Nash extension solution is single-valued. But in our finite bargaining problems, the solution is not always single-valued.

Characterization of Nash Solution

Theorem 4. 1: Efficiency (E), Symmetry (S), Scale Invariance (SI), Contraction Invariance (CI) jointly uniquely determine the Nash solution.

To prove Theorem 4.1, we first prove the following Lemma, Lemma1.

Lemma 1: If a solution F over $\mathcal{B}$ satisfies (E), (S) (CI), and (SI), then for every $x, y \in R^n_+$,

(L1): If $\prod_{i \in N} x_i = \prod_{i \in N} y_i > 0$, then $F(\{0, x, y\}) = \{x, y\}$.

(L2): If $\prod_{i \in N} x_i > \prod_{i \in N} y_i$, then $F(\{0, x, y\}) = \{x\}$.

Proof: Let a solution F over $\mathcal{B}$ satisfies (E), (S) (CI), and (SI). Let $x, y \in R^n_+$.

Suppose first that $\prod_{i\in N} x_i = \prod_{i\in N} y_i > 0$. Noting that $\prod_{i\in N} x_i = \prod_{i\in N} y_i > 0$, it then follows that $x_i > 0,\ y_i > 0$ for all $i \in N$. Define $\alpha \in R^n_+$ as follows: $\alpha_1 = 1$, $\alpha_2 = \frac{x_1}{y_2}\alpha_1, \ldots, \alpha_i = \frac{x_{i-1}}{y_i}, \ldots, \alpha_n = \frac{x_{n-1}}{y_n}\alpha_{n-1}$. Note that, for all $i \in N$, α_i is well defined and $\alpha_i > 0$. Therefore $\alpha \in R^n_{++}$. Furthermore, $\alpha_2 y_2 = \frac{x_1}{y_2} = x_1 = \alpha_1 x_1$, $\alpha_3 y_3 = \frac{x_2}{y_3}\alpha_2 y_3 = \alpha_2 x_2, \ldots, \alpha_n y_n = \frac{x_{n-1}}{y_n}\alpha_{n-1} y_n = \alpha_{n-1} x_{n-1}$, and Noting that $\prod_{i\in N} x_i = \prod_{i\in N} y_i$, $\alpha_n x_n = \frac{x_{n-1}}{y_n}\alpha_{n-1} x_n = \frac{x_n x_{n-1}}{y_n}\alpha_{n-1} = \ldots = \frac{x_n x_{n-1} \ldots x_1}{y_n y_{n-1} \cdots y_2}$, we then have $\alpha_n x_n = y_1 = \alpha_1 y_1$. Consider the permutation π^0 as follows: $\pi^0(i) = i+1$ for $i = 1, \ldots, n-1$ and $\pi^0(n) = 1$. Then $\pi^0(\alpha x) = \alpha y$. That is, αx is a permutation of αy. Let $A \equiv \{\pi(z) \in R^n_+ : z \in \{0, \alpha x, \alpha y\}, \pi \in \Pi\}$. Note that A is symmetric and $x, y, \alpha \in R^n_{++}$. By (E) and (S), it must be true that $F(A) = A \setminus \{0\}$. A straightforward application of (CI) gives $F(\{0, \alpha x, \alpha y\}) = \{\alpha x, \alpha y\}$. By (SI), we then obtain $F(\{0, x, y\}) = \{x, y\}$. Therefore, (L1) is true.

Suppose next that $\prod_{i\in N} x_i > \prod_{i\in N} y_i$. Note that $y \in R^n_+$, we must have $\prod_{i\in N} y_i \geq 0$. Consequently, $\prod_{i\in N} x_i > 0$. Let $\varepsilon \in R^n_+$ be such that $\prod_{i\in N}(y_i + \varepsilon_i) = \prod_{i\in N} x_i$. Clearly, $y + \varepsilon > y$. Consider bargaining problem $B = \{0, x, y, y+\varepsilon\}$. Noting that $x > 0$ and $y + \varepsilon > y$. By (E), $0 \notin F(B)$ and $y \notin F(B)$. By (CI), $F(B) = F(\{0, x, y+\varepsilon\})$. $\prod_{i\in N}(y_i + \varepsilon_i) = \prod_{i\in N} x_i$ and (L1), $F(\{0, x, y+\varepsilon\}) = \{x, y+\varepsilon\}$. So $F(B) = F(\{0, x, y+\varepsilon\}) = \{x, y+\varepsilon)$. A straightforward application of (CI) yields $F(\{0, x, y\}) = \{x\}$. Thus, (L2) is true.

With the help of Lemma 1, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. It can be checked that the Nash solution satisfies (E), (S), (CI), and (SI). We therefore need to show that if a solution satisfies (E), (S), (CI), (SI), then it must be the Nash solution.

Let a solution F over $\mathcal{B}$ satisfies (E), (S), (CI) and (SI). Let $A \in$ Б. We first note that there exists $a \in A$ such that $a_i > 0$ for all $i \in N$. Consequently, $\prod_{i \in N} a_i > 0$. Let $x \in A$ such that $\prod_{i \in N} x_i \geq \prod_{i \in N} y_i$ for all $y \in A$. For any $y \in A$ such that $\prod_{i \in N} y_i < \prod_{i \in N} x_i$, $y \notin F(A)$. For otherwise $y \in F(A)$, , by (CI), $y \in F(\{0, x, y\}$, a contradiction to (L2). Therefore, $F(A)$ must contain at least one $x' \in A$ such that $\prod_{i \in N} x_i' = \prod_{i \in N} x_i$. By (L1), for any $z \in A$ such that $\prod_{i \in N} z_i = \prod_{i \in N} x_i$, z must be in $F(A)$. Since, otherwise, if for some $z \in A$ with $\prod_{i \in N} z_i = \prod_{i \in N} x_i$, $z \notin F(A)$. Then, by (CI), $F(\{0, x', z\} = \{x'\}$, a contradiction to (L1). That is, F must be the Nash solution.

Mariotti (1998) presents a similar characterization of the Nash solution for finite problems using (E), (CI), (SI) and Anonymity, Anonymity is as follows:

Anonymity: For any $A \in$ Б, $F(\pi(A)) = \pi(F(A))$.

Note that Anonymity is a stronger requirement than symmetry.

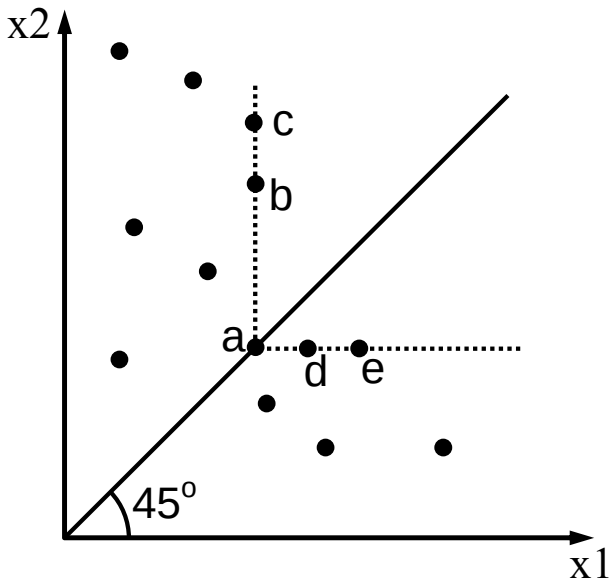
Maximin Solution

The Definitions of Maximin Solution

In this part we discuss the Maximin solution. It is an extension of the Egalitarian solution in standard convex and non-convex bargaining problems to our finite problems. But we can see, in convex and non-convex but comprehensive problems,

the egalitarian solution is always single-valued but in our finite bargaining problems, the Maximin solution is often multi-valued. As illustrated in Figure 12, The Maximin solution to of this two-player bargaining problem is $\{a, b, c, d, e\}$.

Figure 12. Maiximin Solution and Leximin Solution



And the maximin solution is defined as:

Definition 4.2: Maximin solution

A bargaining solution F over $\mathcal{B}$ is the maximin solution if, for all $A \in \mathcal{B}$,

$F(A) = \{a \in A \mid Min_{i \in N}\{a_i\} \geq Min_{i \in N}\{x_i\} \quad \text{for all } x \in A\}\}$.

Characterization of Maximin Solution

Theorem 4.2 The Maximin solution is the unique solution satisfying (WE), (CI), (EI) and (D).

To prove theorem 4.2, we first prove the following lemma, Lemma 2.

Lemma 2. If the solution F over $\mathcal{B}$ satisfies (WE), (CI), (EC), and (D), then for every $x, y \in R^n_+$, and $x_1 = x_2 = \ldots = x_n > 0$,

(L3) if $y > x$ and $\min\{y_1, y_2, \ldots, y_n\} = x_1$, then $F(\{0, x, y\}) = \{x, y\}$.

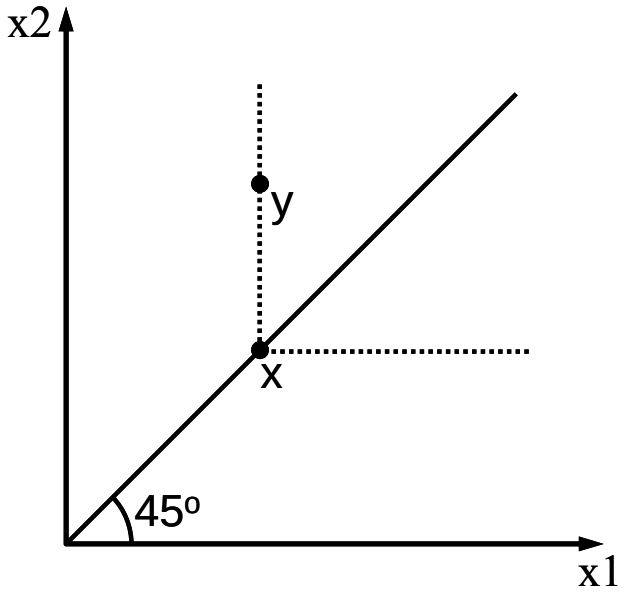
(L4). If $x_1 > \min\{y_1, y_2, \ldots, y_n\}$, then $F(\{0, x, y\}) = \{x\}$.

Proof. Let a solution F over $\mathcal{B}$ satisfies (WE), (CI), (EC), and (D).

Let $x, y \in R^n_+$, and $x_1 = x_2 = \ldots = x_n > 0$.

(i). Suppose first that $y > x$ and $\min\{y_1, y_2, \ldots, y_n\} = x_1$, as illustrated in Figure 13.

Figure 13. Proof of L3

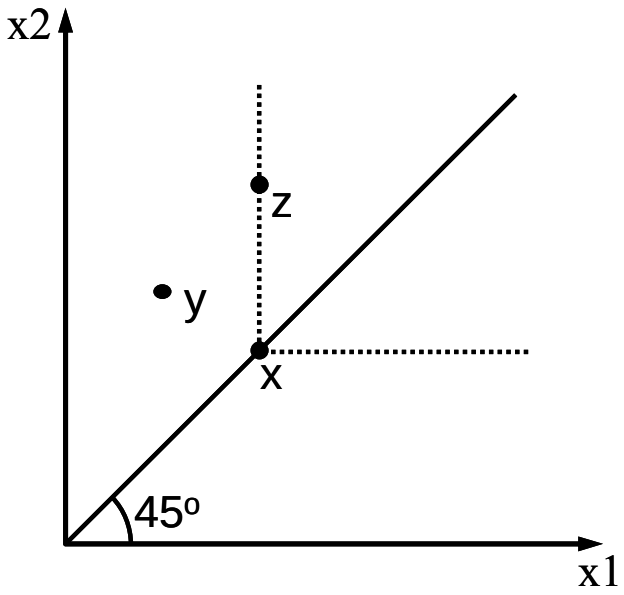


By (WE) and noting that $x >> 0$, it must be true that $F(\{0, x\}) = \{x\}$. By (EC) and noting that $\{0, x\}$ is symmetric, $x \in F(\{0, x, y\})$ follows from that x is weakly efficient on $\{0, x, y\}$. Noting that $y > x$ and $x \in F(\{0, x, y\})$, a straightforward application of (D) and (WE) yields $F(\{0, x, y\}) = \{x, y\}$. Thus, we have shown (L3).

(ii). Next, suppose that $x_1 > \min\{y_1, y_2, \ldots, y_n\}$, then, there exists $z \in R^n_+$ such

that $z >> y, z > x$, and $\min\{z_1, z_2, \ldots, z_n\} = x_1$. As illustrated in Figure 14.

Figure 14. Proof of L4.



From (L3), $F(\{0, x, z\}) = \{x, z\}$. Consider the bargaining problem $\{0, x, y, z\}$. By (WE), clearly, neither 0 nor y is in $F(\{0, x, y, z\})$. If $\{z\} = F(\{0, x, y, z\})$, then by (CI), it would follow that $\{z\} = F\{(0, x, z\})$ which is contradiction with $F(\{0, x, z\}) = \{x, z\}$ which was established a few lines above. Therefore, $x \in F(\{0, x, y, z\})$, which, by (D), (WE) and noting that $z > x$, implies that $F(\{0, x, y, z\}) = \{x, z\}$. Then by (CI), $F(\{0, x, y\}) = \{x\}$. Therefore, (L6) is true.

With Lemma 3, we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. It can be checked that the maximin solution satisfies (WE), (CI), (EC) and (D). We need only to show that if a solution satisfies (WE), (CI), (EC) and (D), then it is the maximin solution.

Let a solution F over $\mathcal{B}$ satisfies (WE), (CI), (EC), and (D). Consider any finite

bargaining problem A from $\mathcal{B}$. Let

$A^* = \{a \in A \mid \min\{a_1, a_2, \ldots, a_n\} \geq \min\{x_1, x_2, \ldots, x_n\}$ for all

$x \in A\}$, $t^* = \min_{a \in A^*}\{a_1, a_2, \ldots, a_n\}$, and $a^* = (t^*, t^*, \ldots, t^*)$. Note that, there exists

$a \in A$ such that $a >> 0$. Hence, $t^* > 0$ and therefore, $a^* >> 0$.

Consider first that $a^* \in A$. If $a^* \neq F(A)$, then for some $x \in A, x \in F(A)$. By (CI), it then would follow that $a^* \neq F(\{0, x, a^*\})$. Note that, x is such that $\min\{x_1, x_2, \ldots, x_n\} \leq a_1^*$. The conclusion, $a^* \neq F(\{0, x, a^*\})$, would be in contradiction with either (L5) or (L6). Therefore, $a^* \in F(\{0, x, a^*\})$. By (D), (CI) and (L5), it then follows that $A^* \subseteq F(A)$. On the other hand, if, for some $x \in F(A)$, $x \notin A^*$, then by (CI), we would have $F(\{0, x, a^*\}) = \{x, a^*\}$. Note however, $x \notin A^*$, so $\min\{x_1, x_2, \ldots, x_n\} < a_1^*$. By (L6), $F(\{0, x, a^*\}) = \{a^*\}$. A contradiction. Therefore, $A^* = F(A)$.

Next, consider that $a^* \notin A$. In this case, we consider the bargaining problem $B = A \cup \{a^*\}$. From above, we can show that $F(B) = A^* \cup \{a^*\}$. Then by (CI), $F(A) = A^*$ follows easily.

Leximin Solution

Definition of Leximin Solution

The Leximin solution maximizes the lexicographic minimum utility a player can get in a problem. As illustrated in Figure 12, the Leximin solution to the problem is $\{c\}$.

Definition 4.4: Leximin solution

A bargaining solution F over B is the leximin solution if, for all $A \in \mathcal{B}$,

$F(A) = \{a \in A \mid a \geq_{lexi\min} x \text{ for all } x \in A\}$.

Recall that $\geq_{lexi\min}$ is defined in definition 2.2.1.

Characterization of Leximin Solution

Theorem 4.4. The Leximin solution is the unique solution satisfying (E), (CI), (S) and (EI).

We use the following lemma, lemma 3, to prove Theorem 4.4.

Lemma 3. If the solution F over $\mathcal{B}$ satisfies (E), (CI), (S), and (EI), then for every $x, y \in R^n_+$, with $x >> 0, y > 0$,

(L5) if $x =_{lexi\min} y$, then $F(\{x, y, 0\}) = \{x, y\}$;

(L6) if $x >_{lexi\min} y$, then $F(\{x, y, 0\}) = \{x\}$.

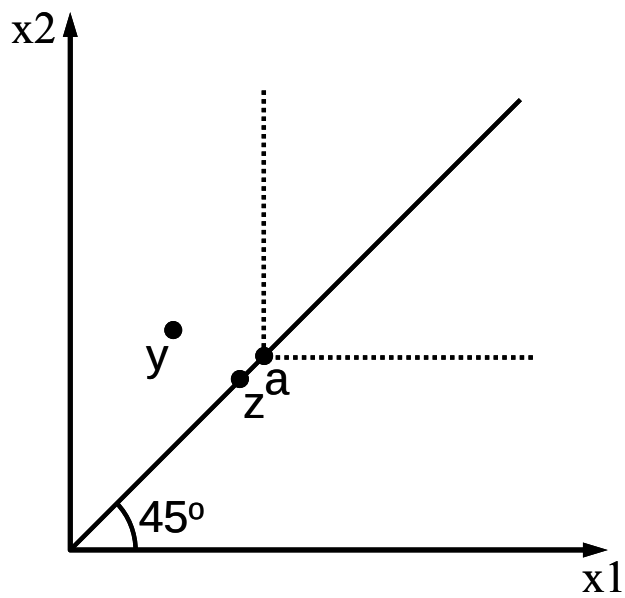
Proof: Let the solution F over $\mathcal{B}$ satisfies (E), (CI), (S), and (EI). Let $x, y \in R^n_+$, with $x >> 0, y > 0$.

Consider first that $x =_{lexi\min} y$, then x must be a permutation of y. Let $A \equiv \{\pi(z) \in R^n_+ : z \in \{0, x, y\}, \pi \in \Pi\}$. By (E), $0 \notin F(A)$. Then, $F(A) = A \setminus \{0\}$ follows from (S) easily. By (CI), it then follows that $F(\{x, y, 0\}) = \{x, y\}$; Thus, (L5) is proved.

Consider next that $x >_{lexi\min} y$, Then for some $k \in \{1, 2, \ldots, n\}$, $(\Theta(x)_1 = \Theta(y)_1, \ldots, \Theta(x)_{k-1} = \Theta(y)_{k-1}, \Theta(x)_k > \Theta(y)_k$. Let $a \in R^n_+$ be such that $a_1 = \Theta(x)_1, a_2 = \Theta(x)_2, \ldots, a_{k-1} = \Theta(x)_{k-1}, a_k = a_{k+1} = \cdots = a_n = \Theta(x)_k$. Note that a is $k-$symmetric and $a >> 0$. By (E), $F(\{0, a\}) = \{a\}$. Noting that a is $k-$symmetric and $a_1 = \Theta(y)_1, a_2 = \Theta(y)_2, \ldots, a_{k-1} = \Theta(y)_{k-1}, a_k = \Theta(x)_k > \Theta(y)_k$, by (EI) we obtain $a \in F(\{0, a, y\})$. We shall show that $a = F(\{0, a, y\})$. Suppose to the contrary

that $a \neq F(\{0,a,y\})$. Given that $a \in F(\{0,a,y\})$, we have $\{a,y\} \in F(\{0,a,y\})$ by (E). Recall that $a_1 = \Theta(y)_1, a_2 = \Theta(y)_2, \ldots, a_{k-1} = \Theta(y)_{k-1}, a_k = \Theta(x)_k > \Theta(y)_k$, there exists $z \in R^n_+$ such that $z_1 = a_1, z_2 = a_2, \ldots, z_{k-1} = a_{k-1}, \Theta(y)_k < z_k = z_{k+1} = \ldots = z_k < a_k$. As illustrated in Figure 4.5. Note that z is $k-$symmetric, and $z >> 0$. By (E) and (EI), $z \in F(\{0,z,y\})$.

Figure 15. Proof of L6



Now we consider the bargaining problem $\{0,a,y,z\}$:

(i) Note that $z < a$, by (E) it must be the case that $0, z \notin F(\{0,a,y,z\}$.

(ii) If $F(\{0,a,y,z\}) = \{a\}$, by (CI) we can get $\{a\} = F(\{0,a,y\})$ a contradiction to our assumption $a \neq F(\{0,a,y\})$.

(iii) If $F(\{0,a,y,z\}) = \{a,y\}$, by (CI), it follows that $\{y\} = F(\{0,z,y\})$, a contradiction to the previous conclusion $z \in F(\{0,z,y\})$.

Form (i), (ii) and (iii), consequently, $F(\{0,a,y,z\}=\phi,$ a contradiction with the non-emptiness of a solution. Therefore, $a=F(\{0,a,y\})$.

If $x=a,$ then we get L(6) directly $F(\{x,y,0\})=\{x\}$.

If $x\neq a,$ we consider the bargaining problem $\{0,a,x,y\}$. Note that $x>a>0$ and $\{a\}=F(\{0,a,y\})$. A straightforward application of (CI) and (E) implies that $\{x\}=F(\{0,a,x,y\})$. By (CI) again, $F(\{x,y,0\})=\{x\}$. Again we get L(6).

Proof of Theorem 4.3. It can be checked that the leximin solution satisfies (E), (CI), (S) and (EI). In what follows, we show that if a solution satisfies (E), (CI), (S) and (EI), then it must be the leximin solution.

Let the solution F over $\mathcal{B}$ satisfies (E), (CI), (S), and (EI). Take any $A\in\mathcal{B}$. Let $A^*=\{a\in A\mid a\geq_{lexi\min} x$ for all $x\in A\}$. We need to show that $F(A)=A^*$. We note that, for all $a,b\in A^*$, $a=_{lexi\min} b$. Let $a^*\in A^*$. If, for some $x\in A, x\in F(A)$ but $x\notin A^*,$ then, by (CI), $x\in F(\{0,a^*,x\}),$ a contradiction with (L6) since $a^*=_{lexi\min} x$. Therefore, $F(A)\subseteq A^*$. Next a straightforward application of (CI) and (L5), $F(A)=A^*$ follows immediately.

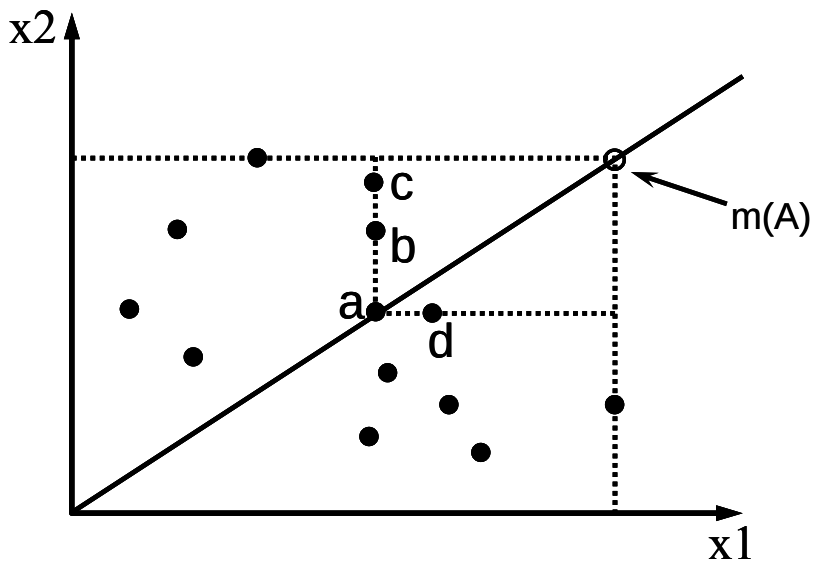
Maxiproportionalmin Solution

Definition of Maxiproportionalmin Solution

In this part we study another series of classic solutions in finite bargaining problem. Kalai and Smorodinsky(1975) came up with their solution which is monotone to the feasible set while it does not satisfy the property of contraction independence. In our finite bargaining problem, the counterpart of Kalai-Smorodinsky solution is Maxiproportionalmin solution defined below. Its lexicographic version is

to be called the Lexiproportinalmin solution which is to be defined in Chapter 4.5.

Figure 16. Maxiproportionalmin Solution and Leximinproportionalmin Solution



Definition 4.4. Maxiproportionalmin Solution: A bargaining solution F over $\mathcal{B}$ is the maxiproportionalmin solution if, for all $A \in \mathcal{B}$,

$$F(A) = \{a \in A \mid \min_{i \in N} \{\frac{a_i}{m_i(A)}\} \geq \min_{i \in N} \{\frac{x_i}{m_i(A)}\} \text{ for all } x \in A\}.$$

As illustrated in Figure 16, the Maxiproportionalmin solution to the problem is $F(A) = \{a, b, c, d\}$

Here we give the definition of the Lexiproportionalmin solution which we will discuss further in next section and we can easily see the difference between these two solutions as illustrated in Figure 16.

Definition 4.5. Lexiproportionalmin Solution: A bargaining solution F over $\mathcal{B}$ is the Lexiproportionalmin solution if, for all $A \in \mathcal{B}$,

$F(A) = \{a \in A \mid (\frac{a_1}{m_1(A)}, \frac{a_2}{m_2(A)}, \ldots, \frac{a_n}{m_n(A)}\} \geq_{lexi\min} (\frac{x_1}{m_1(A)}, \frac{x_2}{m_2(A)}, \ldots, \frac{x_n}{m_n(A)})$ for all $x \in A\}$.

As illustrated in Figure 16, Lexiproportionalmin solution of the problem is $F(A) = \{c\}$.

The Maxiproportionalmin (see also Nagahisa and Tanaka (2002) where they call it the Kalai-Smorodinsky solution for finite bargaining problems) and Lexiproportionalmin solutions can, respectively, be regarded as extensions of the Kalai-Smordinsky and the lexicographic Kalai-Smorodinsky solutions in standard bargaining problems. These two solutions are sensitive to changes in the set of feasible outcomes in a way which the Nash solutions are not. In particular, Maxiproportionalmin and Lexiproportionalmin are sensitive to the ideal point of a bargaining problem. So any change that results in the change of the ideal point will alter the solutions.

Characterization of Maxiproportionalmin Solution

Theorem 4.4. The Maxiproportionalmin solution is the unique solution satisfying (SI), (WE), (WCI), (REC) and (D).

Proof: The necessity of the axioms of theorem 4.4 can be readily checked. To prove the sufficiency part of the theorem, let F be the solution over $\mathcal{B}$ satisfying (SI), (WE), (WCI), (REC) and (D). Let $A \in \mathcal{B}$, Note that $m_i(A) > 0$ for all $i \in N$. Construct the problem $B \equiv \alpha A$ where $\alpha = (\frac{1}{m_1(A)}, \frac{1}{m_2(A)}, \ldots, \frac{1}{m_n(A)})$. Then $m(B) = (1,1,\ldots,1)$. Let F^* denote the maxiproportionalmin solution and $F^{\max i \min}$ the

maximin solution. Then, $F^*(B) = F^{\max i \min}(B)$. In what follows, we shall show that $F(B) = F^{\max i \min}(B)$.

Let $C = \{\pi(b) \in R^n_+ : b \in B \text{ and } \pi \in \Pi\}$. Clearly $m(C) = m(B) = (1,1,\ldots,1)$ and C is symmetric, and $F^{\max i \min}(B) \subseteq F^{\max i \min}(C)$. Consider $a \in R^n_+$ such that $a_1 = a_2 = \ldots = a_n = \max\{\min\{x_1, x_2, \ldots, x_n\} : x \in B\}$. Note that $a >> 0$. For each $i \in N$, let $b^i = (b^i_1, b^i_2, \ldots, b^i_n)$ be such that $b^i_i = a_1$ and $b^i_j = 1$ for all $j \in N \setminus \{j\}$.

Construct the bargaining problem $S \equiv C \cup \{a, b^1, b^2, \ldots, b^n\}$. From the construction, S is symmetric and $m(S) = m(C) = m(B) = (1,1,\ldots,1)$.

By (WE), $F(\{0,a\}) = \{a\}$. Note that $\{0,a\} \subseteq S$, $\{0,a\}$ and S are both symmetric, and a is weakly efficient on S, by (REC), it follows that $a \in F(S)$. Note that, for every $x \in F^{\max i \min}(S)$, it must be true that $x > a$ or $x = a$. By (D), we obtain that $F^{\max i \min}(S) \subseteq F(S)$.

To show that $F^{\max i \min}(S) = F(S)$, we note that, for any $x \notin F^{\max i \min}(S)$, but $x \in S$, from the construction of S, there exists $i \in N$ such that $b^i >> x$. By (WE), it then follows that $x \notin F(S)$. Therefore, $F^{\max i \min}(S) = F(S)$.

Note that $B \subseteq S$, $m(S) = m(B)$ and $B \cap F(S) = B \cap F^{\max i \min}(S) = F^{\max i \min}(B) \neq \phi$. By (WCI), it follows that $F(B) = F^{\max i \min}(B)$. A straightforward application of (SI) yields that $F(A) = F^*(A)$.

In characterizing the maxiproportionalmin solution (what they call the extension of the Kalai-Smorodinsky solution to finite bargaining problem), Nagahisa and Tanaka (2002) use (SI), (S), (WCI), Monotonicity and Continuity. Monotonicity

requires that, for every $A, B \in Б$ with $m(A) = m(B)$ and $A \subseteq B$, if, for some $b \in B$ and every $i \in N$, "b_i is strictly higher than the minimal level of the utilities the solution F gives to the player i under B is strictly higher than that under A.". Continuity requires that any small change in the problem will not lead to large changes in solution outcomes, It is then clear that our axiomatic characterization of the maxiproportionalmin is different from that of Nagahisa and tanaka (2002): in our characterization result, we focus on expansion consistency type properties of the solution, while they focus on monotonicity and continuity aspects of the solution.

Leximproportionalmin Solution

Definition of Leximproportionalmin solution

See definition 4.5. in previous section.

Characterization of Leximproportionalmin Solution

Theorem 4.5. The Leximproportionalmin solution is the unique solution satisfying (SI), (E), (S), (WCI), (WEI) and (IR).

Proof: The necessity of the axioms of Theorem 4.5 can be readily checked. To prove the sufficiency part of Theorem 4.5, let F be a solution over $\mathcal{B}$ satisfying (SI), (E), (S), (WCI), (WEI) and (IR). Let $A \in \mathcal{B}$. Not that $m_i(A) > 0$ for all $i \in N$. Construct $B \equiv \alpha A$ where $\alpha = (\frac{1}{m_1(A)}, \frac{1}{m_2(A)}, \ldots, \frac{1}{m_n(A)})$. Then $m(B) = (1,1,\ldots,1)$. Let F^* denote the leximproportionalmin solution and $F^{lexi\min}$ the leximin solution. Then, $F^*(B) = F^{lexi\min}(B)$. In what follows, we shall show that $F(B) = F^{\max i\min}(B)$.

We first show that $F(B) \subseteq \{x \in B : x \geq_{lexi\min} y, y \in B\} = F^{lexi\min}(B)$.

Let $x^* \in B$ be such that $x^* \geq_{lexi\min} y$ for all $y \in B$. Consider any $z \in B$ such that $x^* \geq_{lexi\min} z$; That is, for some $k \in \{1,2,\ldots,n\}$, $(\Theta(x^*)_1 = \Theta(z)_1, \ldots, \Theta(x^*)_{k-1} = \Theta(z)_{k-1}, \Theta(x^*)_k > \Theta(z)_k$. Consider $a \in R^n_+$ be such that $a_1 = \Theta(x)_1, a_2 = \Theta(x)_2, \ldots, a_{k-1} = \Theta(x)_{k-1}, a_k = a_{k+1} = \cdots = a_n = \Theta(x)_k$. Note that a is $k-$symmetric and $a >> 0$. For each $i \in N$, let $s^i = (s^i_1, s^i_2, \ldots, s^i_n)$ be such that $s^i_i = 1$ and $b^i_j = 0$ for all $j \in N \setminus \{j\}$.

Construct the bargaining problem $C = \{0, a\} \cup \bigcup_{i \in N} \{s_i\}$. Note that $m(C) = m(B) = (1,1,\ldots,1)$. $a >> 0$, By (IR), $F(C) = \{a\}$.

Consider the bargaining problem $D = C \cup \{z\}$. Note that $m(D) = m(C)$, a is $k-$symmetric, $a_1 = \Theta(z)_1, a_2 = \Theta(z)_2, \ldots, a_{k-1} = \Theta(z)_{k-1}$, and a is efficient on D. By (WEI), it follows that $a \in F(D)$. By using a similar argument as in the proof of (L8), we can show that $a = F(D)$. Consider the bargaining problem $B \cup D$. Noting that $m(B \cup D) = m(D) = m(B) = (1,1,\ldots,1)$ and that $z \notin= F(D)$. By (WCI), it then follows that $z \notin= F(B \cup D)$. Again by (WCI), $z \notin= F(B)$.

Therefore, $F(B) \subseteq \{x \in B : x \geq_{lexi\min} y, y \in B\} = F^{lexi\min}(B)$.

To complete the proof of Theorem 4.5, we need to show that $F(B) \supseteq F^{lexi\min}(B)$.

Given the non-emptiness of $F(B)$ and without loss of generality, let $x^* \in F(B)$. Consider the bargaining problem $G = \bigcup_{\pi \in \Pi} \pi(B)$. Note that $m(G) = m(B) = (1,1,\ldots,1)$ and that G is symmetric. Let $y \in F(G)$. Then by (S), $\pi(y) \in F(G)$ for all $\pi \in \Pi$. By the construction, for some $\pi^*(y)$, it must be that $\pi^*(y) \in B$. By (WCI), it then follows that $\pi^*(y) \in F(B)$.

If $x^* >_{lexi\min} y$, it is a contradiction with $F(B) \subseteq F^{lexi\min}(B)$. Therefore, for each $y \in F(G)$, it must be true that $\pi^*(y) \in F^{lexi\min}(B)$. By (S), $F^{lexi\min}(B) \subseteq F(G)$. Then, $F(B) = F^{lexi\min}(B)$ follows from (WCI) immediately.

To summarize, we have shown that $F(B) = F^{lexi\min}(B)$. By (SI), $F(A) = F^*(A)$.

The Independence of Axioms in Lexiproportionalmin Solution

The characterizing axioms in theorem 4.5 are more complicated than other axioms used to characterize other five solutions. Our next result shows that the axioms in Theorem 4.5 are independent. .

Theorem 4.6. The axioms in theorem 4.5 are independent.

Proof: Define the solution $F^1, F^2, F^3, F^4, F^5, F^6$ over $\mathcal{B}$ as follows:

1. (Leximax Proportional solution) A bargaining solution F^1 is defined over $\mathcal{B}$: $F^1(A) = \{a \in A \mid (a_1 / m_1(A), \ldots (a_n / m_n(A)) \geq_{lexi\max} (x_1 / m_1(A), \ldots x_n /(m_n(A))$ for all $x \in A\}$.

$x \geq_{lexi\max} y$ if and only if $[(\Theta_n x > \Theta_n y)$, or $((\Theta_n x = \Theta_n y), (\Theta_{n-1} x > \Theta_{n-1} y))$ or $\ldots$, or $((\Theta_n x = \Theta_n y), \ldots (\Theta_2 x = \Theta_2 y), (\Theta_1 x > \Theta_1 y))$.

2. F^2 is leximin solution defined in definition 4.3.

3. F^3 is maxiproportionalmin solution defined in definition 4.4.

4. We define F^4 as follows: we put some priority on lexiproportionalmin solution, such as: $F(A) \subset$ lexiproportionalmin solution with $F_1(A) \geq F_2(A)$ or $F(A) =$ lexiproportionalmin solution if no x in lexiproportionalmin solution such that $x_1 \geq x_2$.

5. $F^5(A) = \{a \in A \mid (a \text{ is efficient in } A \text{ and } a >> 0\}$.

6. F^6 is Nash solution defined in definition 4.1.

Then we can check that:

1. F^1 satisfies (SI), (E), (S), (WCI), (WEI) but not (IR).

2. F^2 satisfies (IR), (E), (S), (WCI), (WEI) but not (SI).

3. F^3 satisfies (SI), (IR), (S), (WCI), (WEI) but not (E).

4. F^4 satisfies (SI), (E), (IR), (WCI), (WEI) but not (S).

5. F^5 satisfies (SI), (E), (S), (IR), (WEI) but not (WCI).

6. F^6 satisfies (SI), (E), (S), (WCI), (IR) but not (WEI).

From the discussion above we can see, the axioms used to characterize lexiproportinalmin solution are independent with each other and not redundant.

These examples prove Theorem 4.6.

CHAPTER 5.

MAXIMIN AND UTILITARIAN

In this chapter we study the utilitarian solution and other two new solutions which combine the utilitarian solution and maximin solution in different ways.

Utilitarian and Maximin

The utilitarian solution of a bargaining problem selects all those alternatives that have the greatest sum of utilities. The utilitarian principle has a long tradition in moral philosophy and in economics (see, for example, Bentham (1907) and Harsanyi (1955; 1975)).

The Maximin solution of the bargaining problem, as we have studied in chapter 4, consists of all those alternatives at which minimum utility levels are the greatest. This solution was inspired by Rawls (1971; 1974) in his discussion of a theory of justice.

Compared with the Nash solution and the Kalai-Smorodinsky solution, as Shapley (1969) has pointed out, these two solutions both indicate interpersonal comparisons of utilities. In the case of the utilitarian solution, utilities are cardinally comparable, and in the case of the maximin solution, utilities are ordinally comparable.

The utilitarian solution is often criticized that it does not take into account how utilities are distributed among the players. It also treats utility gains by one player

no different from the same amount of utility gains by another player regardless of the status of the players. On the other hand, the maximin solution puts priority to the most unfortunate player (measured in terms of utility levels), while the utility increase of the player who is not the most unfortunate player would not change the outcome of the bargaining.

As we can see, the utilitarian solution and the maximin solution are two extremes: the utilitarian solution is less concerned with the distribution issue while the maximin solution takes the distribution issue to an extreme by focusing on the worst-off player in the game. In this Chapter, we first present a characterization for the utilitarian solution, and then consider two solutions each of which combines the maximin solution and the utilitarian solution in a particular way, and give axiomatic characterization for both.

Definitions

So far the bargaining problems we have studied are the ones that contain at least one element such that every player's utility is strictly greater than the disagreement point $d = 0$. The other kind of bargaining problems such as, $\{0, a\}$ with $a = (0,1)$ is trivial problems for the solutions we have studied in previous section. But the utilitarian solution to this kind of the problems is well defined, so are the Maximin-Utilitarian and Utilitarian-Maximin solutions we will define next. So in this chapter, we extend our bargaining problems to include those that do not contain elements such that every player's utility is strictly greater than the disagreement point. Here we enlarge the family of all finite bargaining problem $\mathcal{B}$ we defined in Chapter 2

to $\mathcal{B}^* = \left\{A \subseteq R_+^n \middle| 0 \in A, A \cap R_+^n \neq \phi \text{ and } 2 \leq \# A < \infty\right\}$.

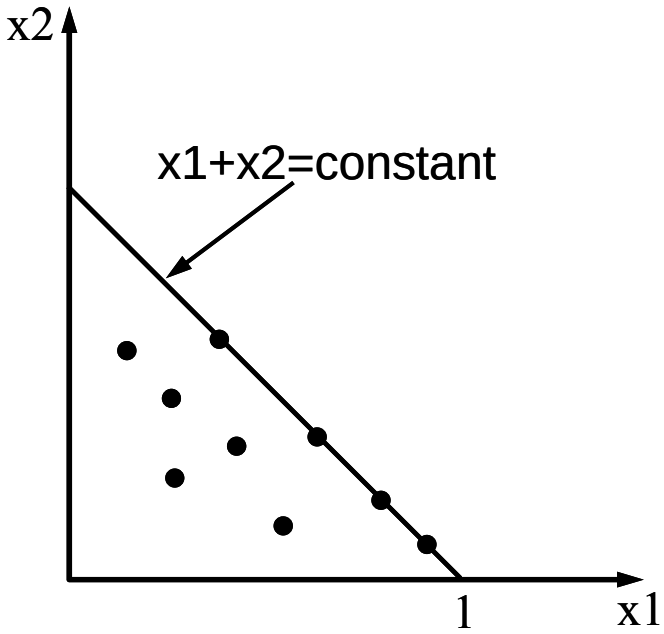
Definition of Utilitarian Solution

The utilitarian solution to a bargaining problem lies on the Pareto-efficient frontier of the problem. Compared with the Nash solution which maximizes the product of player's utilities, the utilitarian maximizes the sum of players' utilities. Figure 17 illustrates the utilitarian solution in a 2-player problem.

Definition 5.1: Utilitarian Solution

A bargaining solution F over $\mathcal{B}^*$ is the utilitarian solution if, for all $A \in \mathcal{B}^*$, $F(A) = \{a \in A \mid \sum_{i \in N} a_i \geq \sum_{i \in N} x_i \text{ for all } x \in A\}$.

Figure 17. Utilitarian Solution



Definition of Maximin-Utilitarian Solution

The Maximin-Utilitarian solution first narrows down the problem to the maximin set of the problem, then from the maximin set of the problem we choose the

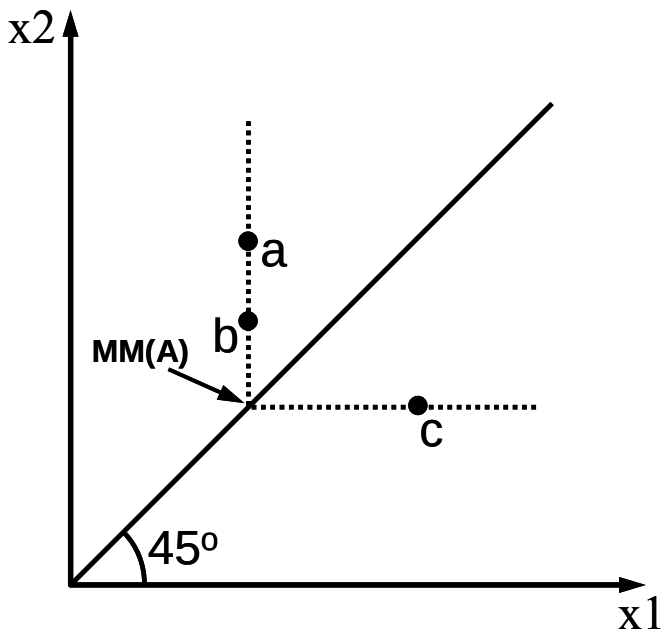
alternatives which maximize the sum of the utilities. Consider a problem A. as illustrated in Figure 5-2 below. The solution to A is given by $F(A) = \{c\}$.

Definition 5.2: Maximin-Utilitarian Solution

A bargaining solution F over $\mathcal{B}^*$ is the Maximin-utilitarian solution if, for all $A \in \mathcal{B}^*$, $F(A) = \{a \in F^{\max i \min}(A) \mid \sum_{i \in N} a_i \geq \sum_{i \in N} x_i$ for all $x \in F^{\max i \min} A\}$.

Where $F^{\max i \min}$ is the maximin solution of the problem.

Figure 18. Maximin-Utilitarian Solution



In a two-player bargaining problem, Maximin-Utilitarian solution is the same as the Leximin solution. However, they are different in the general $n(n > 2)-$player bargaining problems. For example, for the bargaining problem $A = \{0, a, b\}$, in which $a = (1,2,3)$, $b = (1,1.5,4)$, according to the Leximin solution, $F^{lexi \min}(A) = \{a\}$, while according to Maximin-Utilitarian solution, $F^{m-u}(A) = \{b\}$.

Definition of Utilitarian-Maximin Solution

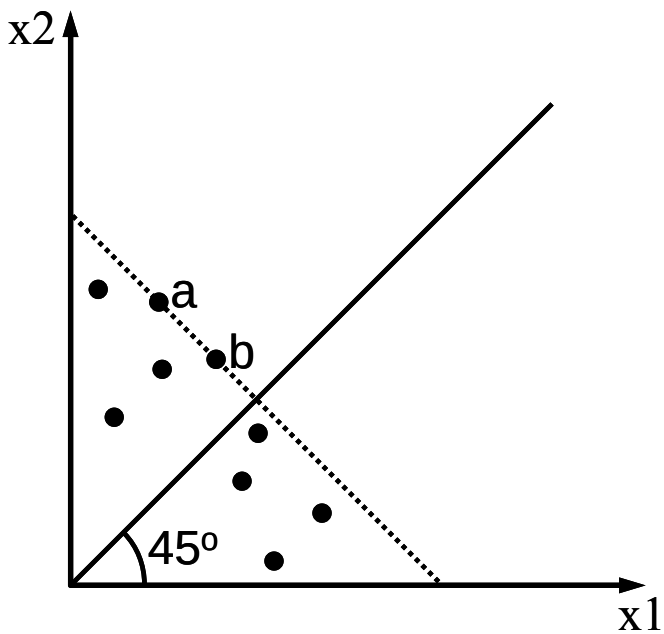
The Utilitarian-Maximin solution, which is different from the Maximin-Utilitarian solution, selects utility vectors from a problem that are the "maximins" of those maximizing the sum of utilities in the problem.

Figure 19 illustrates the Utilitarian-Maximin solution. In the bargaining problem of the figure 19, the utilitarian-maximin solution is given by $F(A) = \{b\}$.

Definition 5.3: Utilitarian-Maximin Solution

A bargaining solution F over $\mathcal{B}^*$ is the Utilitarian-Maximin solution if, for all $A \in \mathcal{B}^*$, $F(A) = \{a \in F^{utilitarian}(A) \mid Min_{i \in N} a_i \geq Min_{i \in N} x_i$ for all $x \in F^{Utilitaian}(A)\}$, where $F^{utilitarian}$ is the utilitarian solution of the problem.

Figure 19. Utilitarian-Maximin Solution



Characterization

Characterization of Utilitarian Solution

Theorem 5.1. The utilitarian solution is the unique solution satisfying (E), (S), (CI) and (PDM).

To establish theorem 5.1, we first prove the following lemma, lemma 4.

Lemma 4. If a solution F over $\mathcal{B}^*$ satisfies (E), (S), (CI) and (PDM), then for every distinct $x, y \in R^n_+$,

(L7). if $x_1 + \ldots + x_n = y_1 + \ldots + y_n$, then $F(\{0, x, y\}) = \{x, y\}$.

(L8). if $x_1 + \ldots + x_n > y_1 + \ldots + y_n$, then $F(\{0, x, y\}) = \{x\}$.

Proof. Let a solution F over $\mathcal{B}^*$ satisfies (E), (S), (CI) and (PDM), Let $x, y \in R^n_+$ be distinct.

Consider first that $x_1 + \ldots + x_n = y_1 + \ldots + y_n = t$. Note that $t > 0$. Let $a^* = (\frac{t}{n}, \ldots, \frac{t}{n})$. If $x = a^*$, then there exists a positive integer p, and $x^1, \ldots, x^p \in R^n_+$ such that x Pigou-Dalton dominates x^1, x^1 Pigou-Dalton dominates x^2, …, x^p Pigou-Dalton dominates y. Let $A \equiv \{\pi(z) \in R^n_+ : z \in \{0, y\}, \pi \in \Pi\}$. By (E) and (CI), we have $F(\{0, y, \pi(y)\}) = \{y, \pi(y)\}$ for any $\pi \in \Pi$. Consider the bargaining problem $B = \{0, y, \pi(y), x^p\}$. Note that x^p Pigou-Dalton dominates y, it then follows that $x^p \in F(B)$. Note that $\{y\} \in F(\{0, y, \pi(y)\})$ so $\{y\} \neq F(\{0, y, \pi(y)\})$. By (PDM), $F(B) \cap F(\{0, y, \pi(y)\}) \neq \phi$. By (CI) it follows that $\{y, \pi(y)\} \subset F(B)$. And with $x^p \in F(B)$ we then have $\{y, \pi(y), x^p\} \subseteq F(B)$. By employing the same argument, we can show that $F(\{0, x, y, \pi(y), x^1, x^2, \ldots. x^p\}) = \{x, y, \pi(y), x^1, x^2, \ldots. x^p\}$. By (CI), it then follows

that $F(\{0,x,y\}) = \{x,y\}$.

If $y = a^*$, we can get the same result.

If $x \neq a^*$ and $y \neq a^*$, by following a similar argument as above, we can show that $F(\{0,x,y,a^*\}) = \{x,y,a^*\}$. By (CI), it follows that $F(\{0,x,y\}) = \{x,y\}$. Thus we get (L7).

Next, consider that $x_1 + \ldots + x_n > y_1 + \ldots + y_n$, Note that, there exists a $z \in R^n_+$ such that $x_1 + \ldots + x_n = z_1 + \ldots + z_n$ and $z > y$. By (E), $y \notin F(\{0,x,y,z\})$. Note that $0 \notin F(\{0,x,y,z\})$. Then $F(\{0,x,y,z\})$ contains either x or z or both. From (L7) and by (CI), it then follows that $\{x,z\} = F(\{0,x,y,z\})$. Therefore, by (CI), $F(\{0,x,y\}) = \{x\}$. This proves (L8).

Proof of Theorem 5.1 It can be checked that the axioms of theorem 5.1 are necessary for the utilitarian solution. We therefore show that if a solution satisfies the axioms of theorem 5.1, it must be the utilitarian solution.

Let F over $\mathcal{B}^*$ satisfies (E), (S), (CI) and (PDM). Take any $A \in \mathcal{B}$. Let $A^* = \{a \in A \mid a_1 + a_2 + \ldots + a_n \geq x_1 + x_2 + \ldots + x_n$ for all $x \in A\}$. Let $a^* \in A^*$ and let $a_1 + a_2 + \ldots + a_n = t$. We show that $F(A) = A^*$. If, for some $x \in A$ with $x_1 + x_2 + \ldots + x_n < t$, $x \in F(A)$, then by (CI), we would have $x \in F(\{0,x,a^*\})$, a contradiction with (L4). Therefore, $F(A) \subseteq A^*$. A straightforward application of (L3) and (CI) then gives us the desired result, $F(A) = A^*$.

Characterization of Maximin-Utilitarian Solution

Theorem 5.2. The Maximin-Utilitarian solution is uniquely determined by (E), (S), (CI), (SEC), (WPDMI).

In order to prove theorem 5.2, first we prove Lemma 5.

Lemma 5. If a solution F over $\mathcal{B}^*$ satisfies (E), (CI), then for every $A \in \mathcal{B}^*$, $x \in R^n_+$, if $\{a\} = F(A)$, $x > a$ then $\{x\} = F(A \cup \{x\})$.

Proof of Lemma 5: Let F over $\mathcal{B}^*$ satisfy (E) and (CI). Let $A \in \mathcal{B}^*$, $x \in R^n_+$, and $\{a\} = F(A)$, with $x > a$. By (E), $a \notin F(A \cup \{x\})$. For any $z \in A$ and $z \neq a$, if $z \in F(A \cup \{x\})$, by (CI), we will get that: $z \in F(A)$, so $z \notin F(A \cup \{x\})$. By the non-emptiness of the solution, $\{x\} = F(A \cup \{x\})$.

Proof of Theorem 5.2: It can be checked that the Maximin-Utilitarian Solution satisfies (E), (S), (WPDMI), (SEC), (CI). We therefore need to show that if a solution satisfies (E), (S), (WPDMI), (SEC), (CI), then it must be the Maximin-Utilitarian Solution.

Let a solution F over $\mathcal{B}^*$ satisfy (E), (S), (WPDMI), (SEC), (CI). First we show that for any $A \in \mathcal{B}^*$, $F(A) \subseteq F^{\max i \min}(A)$.

First consider the problem $B = [A \setminus F^{\max i \min}(A)] \cup mm(A)$. By (E), $mm(A) = F\{0, mm(A)\}$. Notice $mm(A)$ is efficient in B and $\{0, mm(A)\}$ is symmetric, by (EEC), $mm(A) = F(B)$. That means, for any other point $z \in B$, that is, $z \in B \setminus \{mm(A)\}$, that also means $z \notin F^{\max i \min}(A)$, $z \notin F(B)$. Note that $B \subseteq A \cup mm(A)$, then by (CI), $z \notin F(A \cup mm(A))$ for any $z \notin F^{\max i \min}(A)$,

Then if

(i). $mm(A) \in A$, then $A \cup mm(A) = A$, it follows $z \notin F(A)$ for any $z \notin F^{\max i \min}(A)$,

(ii) $mm(A) \notin A$, then there must be some $b \in A$ such as $b_i = mm_i(A)$ for some

$i \in N$ and $b_j > mm_j(A)$ for other $j \neq i$. That is, $b > mm(A)$ but not $b >> mm(A)$.

By Lemma 5 we get $F\{B \cup \{b\}) = b$. Then by (CI), $F([\{B / mm(A)] \cup \{b\}) = b$. That is, $z \notin F(\{[B / mm(A)] \cup \{b\})$ for any $z \notin F^{\max i \min}(A)$, Noting that $[B / mm(A)] \cup \{b\}] \subset A$, then we can get $z \notin F(A)$ for any $z \notin F^{\max i \min}(A)$,

From (i) and (ii), we get the result that if $z \notin F^{\max i \min}(A)$, then $z \notin F(A)$. that means, $F(A) \subseteq F^{\max i \min}(A)$.

By the same reasoning as in the proof of Theorem 5.1, and noting that during the same time the center point of the problems remains unchanged, we can show that $F(A)$ is the utilitarian solution in $F^{\max i \min}(A)$.

Characterization of Utilitarian-Maximin Solution

Theorem 5.3: The Utilitarian-Maximin solution is uniquely determined by (E), (S), (CI), (WPDMII), (EX) and (RI).

Proof of Theorem 5.3. It can be checked that the Utilitarian-Maximin Solution (E), (S), (CI), (WPDMII), (EX), (RI). We therefore need to show that if a solution satisfies (E), (S), (CI), (WPDMII), (EX) and (RI), then it must be the Utilitarian-Maximin Solution.

Let a solution F satisfy (E), (S), (CI), (WPDMII), (EX) and (RI).

Step 1. We show that $F(A) \subseteq F^{utilitarian}(A)$, that is, if there exist $x, y \in A$, such that $x_1 + \ldots + x_n < y_1 + \ldots + y_n$, then $x \notin F(A)$.

Suppose to the contrary that for some x in A such that $x \in F(A)$, and there is $y \in A$, with $x_1 + \ldots + x_n = t < y_1 + \ldots + y_n = s$. Then, consider a, such that $a_1 = a_2 = \ldots = a_n = \frac{t}{n}$. We can check that either $a = x$ or a Pigou-Dalton

dominates x, then By (WPDMII),

$a \in F(A \cup \{a\})$. (*).

(i) If $\min_{i \in N} y_i \geq a_i$, noting that $a_1 = a_2 = \ldots = a_n$ and $\sum_{i \in N} y_i > \sum_{i \in N} a_i$, we can get that $y > a$. By (E), $a \notin F(A \cup \{a\})$. a contradiction to (*). In this case, we get the conclusion that $x \notin F(A)$.

(ii) If $\min_{i \in N} y_i < a_i$, noting that $a_1 = a_2 = \ldots = a_n$ and $\sum_{i \in N} y_i > \sum_{i \in N} a_i$, there exists $z \in R_+^n$, such that $\sum_{i \in N} z_i = \sum_{i \in N} y_i$, and for some $0 < k < N$, $\Theta(z)_1 = \Theta(z)_2 = \ldots = \Theta(x)_k = a_i$, $\Theta(z)_{k+1} > a_i$. And note that z Pigou-Dalton dominates y and $mm(\{0, a\}) = mm(\{0, a, z\})$. Then by RI, $a \in F([A \cup \{a\} \cup \{z\}] / y)$. But $z > a$, a contradiction with (E).

Therefore if $x, y \in A$, if $x_1 + \ldots + x_n < y_1 + \ldots + y_n$, then $x \notin F(A)$. That is, $F(A) \subseteq F^{utilitarian}(A)$.

Step 2. We prove that for all $x, y \in F^{utilitarian}(A)$, if $\min_{i \in N} x_i < \min_{i \in N} y_i$, then $x \notin F(A)$.

Consider the problem $B = \{0, x, y\}$, in which $x_1 + \ldots + x_n = y_1 + \ldots + y_n$, and $\min_{i \in N} x_i < \min_{i \in N} y_i$. By (E), it must be true that $F(B) \subseteq \{x, y\}$. If $\{x\} \subseteq F(B)$, and noting that $x = F(\{0, x\})$, we have $\{x\} = F(B) \cap F(\{0, x\})$. Note that $mm(B) \neq mm(\{0, x)\}$. From $\min_{i \in N} x_i < \min_{i \in N} y_i$, it follows that $F(B) \cap F(\{0, x\}) = \phi$, a contradiction. So we get $x \notin F(B)$. By (CI), $x \notin F(A)$.

Step 3. We prove that F over $\mathcal{B}^*$ is the Utilitarian-Maximin solution.

Form (step 1) and (step 2) we get that

$F(A) \subseteq \{a \in F^{utilitarian}(A) \mid Min_{i \in N} a_i \geq Min_{i \in N} x_i$ for all $x \in F^{Utilitaian} A\}$. By the same

reasoning as in the proof of theorem 5.1, we can get the conclusion that the Utilitarian-Maximin solution is uniquely determined by (E), (S), (CI), (WPDMII), (EX) and (RI).

The Independence Property of the Axioms

In this section, we show that there is no superfluous axiom in the characterization of the Utilitarian, Maximin-Utilitarian and Utilitarian-Maximin solutions.

The Independence of the Axioms in Utilitarian Solution

Theorem 5.4. The axioms used in theorem 5.1, (E), (S), (CI), (PDM) are independent.

We will use the same notation to define the alternative solutions as in theorem 4.6 in the following proofs.

Proof: Define the solution F^1, F^2, F^3, F^4 over Б as follows:

1. A bargaining solution F^1 is defined over $\mathcal{B}^*$: $F^1(A) = \{0\}$ for all $A \in \mathcal{B}^*$. That is, F^1 always chooses the disagreement point of any problem.

2. F^2 is defined over $\mathcal{B}^*$ as follows: for all $A \in \mathcal{B}^*$.

$F^2(A) = F^{Utilitarian}(A)$ if $F^{Utilitarian}(A) = \{(0,0,\ldots,0,l)\}$;

$F^2(A) = (F^{Utilitarian}(A))/\{(0,0,\ldots,l)\}$ if $F^{Utilitarian}(A) \neq \{(0,0,\ldots,0,l)\}$.

3. F^3 is defined over $\mathcal{B}^*$ as: for all $A \in \mathcal{B}^*$,

$F^3(A) = \{x \in F^{Utilitarian}(A) \mid$ there is no $y \in A$ such that x is Pigou-Dalton dominated by y or there is $z \in A$ such that $x = \pi(z)$, for some $\pi \in \Pi.\}$.

4. F^4 is defined as as the Nash solution.

Then we can check following:

1. F^1 satisfies (S), (CI),(PDM) but violates (E).

2. F^2 says, if there is a point which is on the last axis, and it also belongs to the utilitarian solution, if the utilitarian solution only contains this point, we will choose this point. If the utilitarian solution of the problem includes other points than this one, we will choose all other points in the utilitarian solution of the problem excluding this point. This solution does not satisfy the symmetry axiom and it is also easy to check that this solution satisfies (E), (CI).

Now we check for (PDM). In the first condition, $F^2(A) = F^{Utilitarian}(A)$, $F^2(A) = F^{Utilitarian}(A)$ satisfies (PDM). In the second condition, $F^{Utilitarian}(A) \neq \{(0,0,\ldots,0,l)\}$. If $(0,0,\ldots,0,l) \notin F^{utilitarian}(A)$, then $F^2(A) = (F^{Utilitarian}(A))$ satisfies (PDM). If $(0,0,\ldots,0,l) \in F^{utilitarian}(A)$, but we can see there is no $x \in F^{utilitarian}(A)$ such that $(0,0,\ldots,0,l)$ PDM x. So $(0,0,\ldots,0,l) \notin F^2(A)$ does not violate (PDM).

From above we can get, F^2 satisfies (E), (CI), (WPD) but violates (S).

3. For any problem $A \in \mathcal{B}^*$, $x \in F^{Utilitarian}(A)$ and if there is $y \in A$ such that x is Pigou-Dalton dominated by y and there is $z \in A$ such that $x = \pi(z)$, $\pi \in \Pi$. If y is the only symmetric point of x and we delete y from the problem A and x is not in the solution to the new contracted problem. So F^3 satisfies (E), (S), (PDM) but violates (CI).

4. F^4 satisfies (E), (S), (CI), but violates (PDM).

The Independence of the Axioms in Maximin-Utilitarian Solution

Theorem 5.5. The axioms used in theorem 5.2, (E), (S), (CI), (SEC), (WPDMI), are independent.

Proof: Define the solution F^1, F^2, F^3, F^4, F^5 over Б as follows:

1. A bargaining solution F^1 is defined as the Maximin solution over $\mathcal{B}$*.

2. F^2 is defined over $\mathcal{B}$ as: for all $A \in \mathcal{B}$*.

$F^2(A) = F^{M-U}(A)$ if $F^{M-U}(A) = \{(0,0,\ldots,0,l)\}$;

$F^2(A) = (F^{M-U}(A))/\{(0,0,\ldots,l)\}$ if $F^{M-U}(A) \neq \{(0,0,\ldots,0,l)\}$.

There F^{M-U} is the Maximin-Utilitarian solution defined in definition 5.2.

3. F^3 is defined over $\mathcal{B}$* as: for all $A \in \mathcal{B}$*,

$F^3(A) = \{x \in F^{M-U}(A) \mid$ there is no $y \in A$ such that x is Pigou-Dalton dominated by y or there is $y \in A$ such that $x = \pi(y),\ \pi \in \Pi.\}$.

4. We define F^4 as the Utilitarian solution in definition 5.1.

5. We define F^5 as the Leximin solution in definition 4.4.

Then we can check:

1. F^1 satisfies (S), (CI),(SEC), (WPDMI) but violates (E).

2. F^2 satisfies (E), (CI), (SEC), (WPDMI(but violates (S).

3. F^3 satisfies (E), (S), (SEC), (WPDMI) but violates (CI).

4. F^4 satisfies (E), (S), (CI), (WPDMI) but violates (SEC).

5. F^5 satisfies (E), (S), (CI), (SEC) but violates (WPDMI).

The Independence of Axioms in Utilitarian-Maximin Solution

Theorem 5.6. The axioms used in theorem 5.3, (E), (S), (CI), (WPDMII), (EX)

and (RI) are independent with.

1. A bargaining solution F^1 is defined as the Maximin solution over $\mathcal{B}^*$.

2. F^2 is defined over $\mathcal{B}^*$ as: for all $A \in \mathcal{B}^*$.

$F^2(A) = F^{U-M}(A)$ if $F^{U-M}(A) = \{(0,0,\ldots,0,l)\}$;

$F^2(A) = (F^{U-M}(A))/\{(0,0,\ldots,l)\}$ if $F^{U-M}(A) \neq \{(0,0,\ldots,0,l)\}$.

There F^{U-M} is the Utilitarian-Maximin solution defined in definition 5.3.

3. F^3 is defined over $\mathcal{B}^*$ as: for all $A \in \mathcal{B}^*$,

$F^3(A) = \{x \in F^{U-M}(A) \mid$ there is no $y \in A$ such that x is Pigou-Dalton dominated by y or there is $y \in A$ such that $x = \pi(y),\ \pi \in \Pi.\}$.

4. (Utilitarian-Leximin solution). A bargaining solution F^4 over $\mathcal{B}^*$ is the Utilitarian-Maximin solution if, for all $A \in \mathcal{B}^*$, $F(A) = \{a \in F^{utilitarian}(A) \mid a \geq x$ for all $x \in F^{Utilitaian}(A)\}$. In which $F^{utilitarian}$ is the utilitarian solution of the problem.

5. We define F^5 as the Utilitarian solution in definition 5.1.

6 We define F^5 as the Maximin-Utilitarian solution in definition 5.2.

Then we can check:

1. F^1 satisfies (S), (CI), (WPDMII), (EX), (RI) but violates (E).

2. F^2 satisfies (E), (CI), (WPDMII), (EX), (RI) but violates (S).

3. F^3 satisfies (E), (S), (WPDMII), (EX), (RI) but violates (CI).

4. F^4 satisfies (E), (S), (CI), (RI), (EX), (RI) but violates (WPDMII).

5. F^5 satisfies (E), (S), (CI), (WPDMII), (RI) but violates (EX).

6. F^5 satisfies (E), (S), (CI), (WPDMII), (EX) but violates (RI).

CHAPTER 6.

EGALITARIAN SOLUTION

In this chapter we discuss some problems in the egalitarian solution in finite bargaining problems. For a comprehensive problem, whether the problem is convex or non-convex, the egalitarian solution to that problem is the one which assigns equal utility to each player. But for a finite bargaining problem, the egalitarian solution is more complex.

Egalitarian Solution in Finite Problems

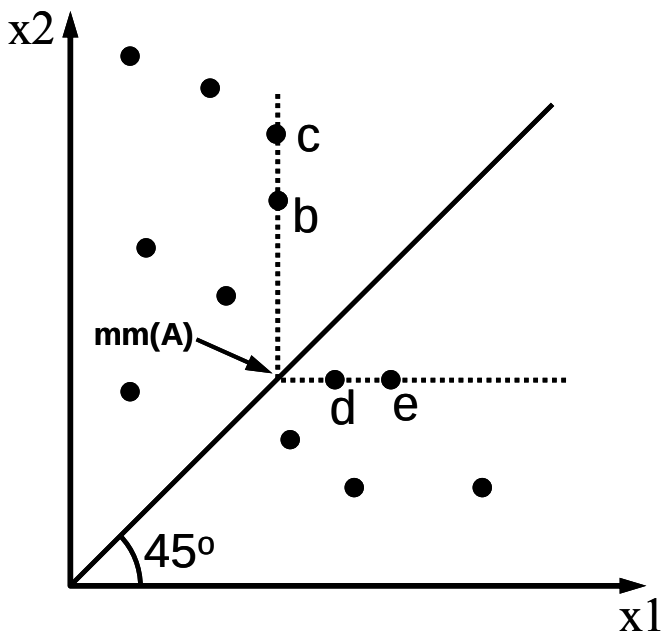
In bargaining problems with comprehensive and compact feasible sets, including convex and non-convex problems, the egalitarian solution is defined as the alternative that assigns to each player the same level of utility and that is weakly efficient. In our finite bargaining problems, such an alternative may not exist for some problems. The Maximin solution and Leximin solutions can be regarded as approximations to the Egalitarian solution which is originally defined for comprehensive problem.

But when we observe the Maximin solution and Leximin solution closely, either of them is not the closest approximations to the egalitarian solution for finite problems. For example, as illustrated in Figure 20, we may want to argue that $\{d\}$ should be the solution to the problem A since d is the closest point to $mm(A)$. If $mm(A)$ is available in the problem, $mm(A)$ should be the solution to the problem. If it is not available, the alternative which is closet to it should be the solution, which

in the problem illustrated in Figure 20, is d . Also, in alternative d , the difference between utilities that the two players get respectively is smallest, which is the gist of the Egalitarian solution.

In Figure 20, we have $b > d, c > d, e > d$, but none of b, c, e is in the solution because the distance between $mm(A)$ and any of b, c, e is greater than the distance between $mm(A)$ and d . respectively. And also the differences between the utilities of the two players in those alternatives, for example, for alternative b , the difference between the utilities of the two players is: $| b_1 - b_2 |$, are all greater than the difference in alternative d, which is: $| d_1 - d_2 |$

Figure 20. The Egalitarian Solution in Finite Bargaining Problem



In what follows, we shall illustrate several versions of the egalitarian solution

for finite problems.Variants of the Egalitarian Solution.

Variants of the Egalitarian Solution

We discuss several variants of the egalitarian solution for finite problems. A basic feature of our proposed solution is the following. Alternatives in a solution to a bargaining problem should be those that are "closest" to the center point and are weakly efficient; or that are "closest" to the center point among all the alternatives in the maximin solution; or that exhibit the "least" degree of inequality in terms of utility levels among players and are weakly efficient. To define our solutions formally, we start by discussing the distance between two utility vectors.

The Definition of Distance

The "closeness" of two utility vectors is typically measured in terms of the distance between them. Given a distance function, the closer the two alternatives are, the shorter the distance between them is. There are various ways to measure the distance between two alternatives. Consider the following examples: For any $x, y \in R_+^n$,

(1) The sum distance is defined as $d(x,y) = \sum_{i \in N} | x_i - y_i |.$,

(2) The max distance is defined as: $d(x,y) = \max_{i \in N} | x_i - y_i |$.

In general, a function d is considered as a distance function if it has the following properties: for any $a, b, c \in R_+^n$,

1. $d(a,a) = 0$.
2. $d(a,b) = d(b,a)$.
3. $d(a,c) \le d(a,b) + d(b,c)$.

Clearly, the sum and the max distance functions satisfy the above requirements for a general distance function.

Egalitarian Solution in Two-Player Problems

Definition of Egalitarian Solution in 2-player Problems

First we define the Egalitarian solution in 2-player problems which chooses the alternatives which is closest to the center point and which is weakly efficient.

Definition 6.1. Type I Egalitarian Solution in 2-player problems. A bargaining solution F over $\mathcal{B}$ is the Egalitarian Solution I if, for all $A \in \mathcal{B}$, $F(A) = \{a \in A \mid d(a, mm(A) \leq d(x, mm(A)$ for all $x \in A$ and there is no other $x \in A$ such that $x >> a$ $\}$.

The distance function can be any one defined in last section. In this two-player bargaining model, we can check all of the distance functions give us the same result.

Second, we confine our attention to the alternative which is closest to the center point among those belonging to the maximin solution. In a 2-player problem, the measurement of the distance between the alternative in the maximin and the center point coincides with each other and can be expressed as: $d(x, mm(A)) = |x_1 - x_2|$, . Since $x \in \max i \min(A)$, $\min(x_1, x_2) = mm_i(A)$. Then the second type of the definition of egalitarian solution in 2-player problems is defined as in Definition 6.2 below.

Definition 6.2. Type II Egalitarian solution for 2-player bargaining problems: When $n = 2$, the type II egalitarian solution is defined as follows: For any $A \in \mathcal{B}$, $F(A) = \{a \in \max i \min(A) \| a_1 - a_2 | \leq | x_1 - x_2 |\}$ for

all $x \in \max i \min(A)\}$.

Thirdly, we define the solution which chooses the alternatives which have the smallest difference between the player's utilities and which are weakly efficient. For a 2-player game, the measurement of the difference between the utilities of two players is given by, $|x_1 - x_2|$. So the definition of the type III Egalitarian solution is as follows:

Definition 6.3. Type III Egalitarian Solution for 2-player bargaining problems: When $n = 2$, the egalitarian solution is defined as follows:

For any $A \in \mathcal{B}$, $F(A) = \{a \in A \| a_1 - a_2 \| \leq \| x_1 - x_2 \|$ for all $x \in A$ and there is no $x \in A$ such that $x >> a.\}$

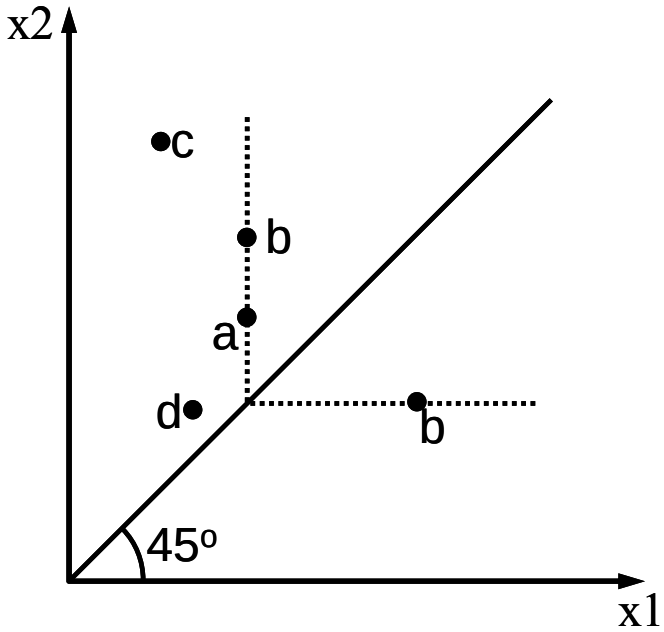
It is easy to verify that the three types of the Egalitarian solution defined above coincide with each other for 2-player bargaining problems. As a consequence, in the following discussion concerning 2-player bargaining problems, we will not make a distinction and will use the egalitarian solution to refer to all of them.

Characterization of Egalitarian Solution in 2-player Problem

Theorem 6.1: (WE), (S), (CI), (HE) uniquely determine the egalitarian solution for 2-Player bargaining problems.

Proof of Theorem 6.1. It can be checked that the Egalitarian solution for two-player bargaining problems satisfies (WE), (S), (CI), (HE). We therefore need to show that if a solution satisfies (WE), (S), (CI), (HE), then it must be the Egalitarian Solution for two-player bargaining problems.

Figure 21. Illustration of the Proof of Theorem 6.1



Let a solution F satisfy (WE), (S), (CI), and (HE) and let $A \in \mathcal{B}$. Define $B \equiv \{\pi(a) \in R_+^2 : a \in A, \pi \in \Pi\}$. From Figure 6.2, we can see that, for any a belonging to Egalitarian solution of B, and any other alternative $x \neq a$, either $x << a$, like point d, or $x_i \leq a_i \leq a_j < x_j$ or $x_i \leq a_i < a_j \leq x_j$ or $x_i < a_i \leq a_j \leq x_j$ like point b, c. then by the non-emptiness of the solution and (HE), (WE), $F(B) \subseteq$ egalitarian solution of the problem B. By S, any a belonging to the Egalitarian solution of $B \subseteq F(B)$. Therefore $F(B) = \{a, \pi(a), \pi \in \Pi\}$ follows from (HE). By (CI), $F(A)$ is given by the Egalitarian solution.

Egalitarian Solutions in More-Than-Two-Player Problems

The egalitarian solution is straightforward and unambiguous in a two-player bargaining problem. But in a problem with more than two players, according to type III definition, we need to find a point which is on the weak efficient frontier and

which has the smallest difference of the utilities of all players. But when $n > 2$, how can we tell the difference of the utilities that all the players get is smaller or bigger? For example, consider the following two alternatives: $(1,4,6,8)$ and $(2,3,5,9)$, . How can we tell which point has the bigger difference?

Also in a problem with more than two players, the type I solution and type II solution are different in general. For example, for a problem $A = \{0, a, b\}$ with $a = (1,2,3)$ and $b = (0.9,2.1,2.2)$, if we use the max distance, we can see, according to type I egalitarian solution, $F(A) = \{b\}$, while according to type II egalitarian solution, $F(A) = \{a\}$.

In the next section we focus on the type I (which confines to the weakly efficient alternatives in the problem) and type II (which confines to the alternatives in the Maximin solution of the problem) Egalitarian solution in the bargaining problems with more than two players. From the following definitions we can see, in the bargaining problems with more than two players, not only type I and type II solutions are different, but with different distance definitions, we can get different definitions of Egalitarian solutions from the same type as well.

Type I Egalitarian Solution

As defined in definition 6.1, type I Egalitarian solution is the alternatives which are "closest" to the center point among all weakly efficient alternatives. By using different distance functions, we can define various type I Egalitarian solutions below.

Definition 6.4. Type I Egalitarian Solution with sum distance. A bargaining solution F over $\mathcal{B}$ is the Egalitarian Solution I if, for all $A \in \mathcal{B}$,

$F(A) = \{a \in A \mid d(a, mm(A) \leq d(x, mm(A))$ for all $x \in A$ and there is no other $x \in A$ such that $x >> a$ $\}$.

The distance functions are discussed in *definition of distance section*.

Type II Egalitarian Solution

Type II Egalitarian solution is defined as the alternatives which are "closest" to the center point among all the alternatives in the Maximin solution of the problem. Also, we can define various solution using different distance functions.

Definition 6.5. Type I Egalitarian Solution with sum distance. A bargaining solution F over $\mathcal{B}$ is Egalitarian Solution I with sum distance if, for all $A \in \mathcal{B}$, $F(A) = \{a \in F^{\max i \min}(A) \mid \sum_{i \in N} a_i \leq \sum_{i \in N} x_i$ for all $x \in F^{\max i \min}(A)\}$.

The distance behind this definition is $d(x, y) = \sum_{i \in N} | x_i - y_i |$. To see this, we note that we are confined to the alternatives which belong to the maximin solution, the center point of a problem A, is $y = mm(A)$. ; then the minimization problem $Min : d(x, y)$ where x belongs to the Maximin solution of the problem is the same as $Min : \sum_{i \in N} | x_i - y_i | = \min : \sum_{i \in N} x_i$.

Definition 6.6. Type II Egalitarian Solution with max distance. A bargaining solution F over $\mathcal{B}$ is Egalitarian Solution with max distance if, for all $A \in \mathcal{B}$, $F(A) = \{a \in F^{\max i \min}(A) \mid Max_{i \in N} \{a_i\} \leq Max_{i \in N} \{x_i\}$ for all $x \in F^{\max i \min} A\}$.

The distance behind this definition is $d(x, y) = \max_{i \in N} | x_i - y_i |$. Note that the distance between any $x \in F^{\max i \min}(A)$ and $mm(A)$ is $\max_{i \in N} \{x_i - \alpha_i\}$ $= \max_{i \in N} \{x_i\} - \alpha_i$. So the minimization problem $\min : d(x, mm(A))$ is the same as $\min : \max_{i \in N} \{x_i\}$.

Definition 6.7. Type II Egalitarian Solution with general distance function. A bargaining solution F over $\mathcal{B}$ is the Egalitarian Solution with a general distance function d if, for all $A \in \mathcal{B}$, $F(A) = \{a \in F^{\max i \min}(A) \mid d(a, mm(A) \leq d(x, mm(A)$ for all $x \in F^{\max i \min} A\}$.

Additional Properties Used in This Section

Axiom 6.1: Strong expansion Consistency (SEC): For each $A, B \in \mathcal{B}$ with $A \subseteq B$, if A is symmetric, $\{a\} = F(A)$ and a is weakly efficient on B, then $\{a\} = F(B)$.

Strong expansion Consistency is another type of expansion property similar to expansion consistency described in Axiom 3.9. It requires that, given a symmetric problem A that has a as the unique solution, if A is enlarged to B so that B is a superset of A and if a continues to be weakly efficient on B, then a should still be the unique solution of B. The difference between SEC and EC is that after the enlargement, SEC requires that $\{a\}$ is still the solution to the new problem while EC requires that that a is contained in the solution to the enlarged problem.

Axiom 6.2: Substitution Property (SP): For any symmetric $A \in \mathcal{B}$ with $F(A) = \{a\}$, for $x \in R^n_+$, x is weakly efficient in $A \cup \{x\}$, then if $d(x,a) < d(y,a)$ for all $y \in A/\{a\}$ $A \Rightarrow$ $\{x\} = F(A \cup \{x\}/a)$; and if $d(x,a) \leq d(y,a)$ for all $y \in A/\{a\} \Rightarrow$ $x \in F(A \cup \{x\}/a)$.

Axiom 6.2 is straightforward. If we substitute the unique solution in a symmetric problem with an alternative which is weakly efficient in the new problem and also is closer to the original solution than any other alternatives from the original

problem, then this substitution should be the only alternative chosen by the solution to the new problem; if that substitution is one of the alternatives which are closest to the original solution, it then belongs to the solution to the new problem.

Characterization of Type I Egalitarian Solution in More-Than-Two-Player Problems

Theorem 6.2: The Type I Egalitarian solution is uniquely determined by (WE), (S), (CI), (SEC), (SP).

Proof of Theorem 6.2. It can be checked that the Egalitarian Solution satisfies (WE), (S), (CI), (SEC), (SP). We therefore need to show that if a solution F satisfies(WE), (S), (CI), (SEC), (SP), then it must be the Egalitarian Solution.

For any $A \in \mathcal{B}$, first let $a = mm(A)$. For a bargaining problem $\{0, a\}$, by (WE), $\{a\} = F(A)$. Note that $\{0, a\}$ is symmetric and $\{0, a\} \subseteq A \cup \{a\}$, also a is weakly efficient in $A \cup \{a\}$, by (SEC) we get that $a = F(A \cup \{a\})$.

(i). If $a \in A$, then $a = F(A)$. we get the conclusion.

(ii). If $a \notin A$, let $B = \{z \in A \mid z$ is closest to a in A and no $x \in A$ such that $x >> a\}$. then B is type I egalitarian solution of the problem.

We want to show that $B = F(A)$.

First we show that $F(A) \subseteq B$.

Let $C = A / B$. Construct the symmetric bargaining problem $D = \{a\} \cup \{\pi(x) : x \in C, \pi \in \Pi\}$. By the same reasoning as above, $a = F(D)$.

Noting that D is symmetric and for any $z \in B$, z is weakly efficient in $D \cup \{z\}$, z is closer to a than any other alternative in $D / \{a\}$, then by (SP), we can

get $z = F((D \cup \{z\})/\{a\})$.

Noting that $[(D \cup \{z\})/\{a\}] \cap A = \{z\} \cup C$, that is,

$\{z\} \cup C \subseteq (D \cup \{z\})/\{a\}$, and by (CI), we can get $\{z\} = F(\{z\} \cup C)$.

That is, for any $z \in B$, $\{z\} = F(\{z\} \cup C)$, where $C = A/B$. Then if there is an

$x \in F(A)$ and $x \notin B$, that is, $x \in F(A)$ and $x \in C$, then by CI, we can get

$x \in F(\{z\} \cup C)$, it is a contradiction to $\{z\} = F(\{z\} \cup C)$. Therefore if $x \in F(A)$,

$x \in B$.

Next, we show that $B \subseteq F(A)$.

For any $z \in B$, let $X = A/\{z\}$. Construct a symmetric problem

$Y = \{a\} \cup \{\pi(x) : x \in X, \pi \in \Pi\}$. By the same reasoning as above, $a = F(Y)$. Noting

that Y is symmetric and for $z \in B$, z is weakly efficient in $Y \cup \{z\}$, z is among the

closest to a in the alternative in $Y/\{a\}$,, then by (SP), we can get

$z \in F((Y \cup \{z\})/\{a\})$. Noting that $A \subseteq F((Y \cup \{z\})/\{a\})$, then by (CI) we can get

$z \in F(A)$.

CHAPTER 7.

CONCLUSIONS AND CONTRIBUTIONS

In this dissertation, we have extended the standard bargaining model to the one that consists of all finite bargaining problems. First we define and characterize various solutions which extend their counterparts in convex and non-convex but comprehensive problems. Then we study other solutions which are new and unique in finite problems. Finally, we present several variants of the Egalitarian solution for finite bargaining problem.

From the first part of our work, we note that the standard extensions of the Nash's original axioms for convex problems to our problem continue to give the characterization result of the Nash solution for finite bargaining problems. The behaviors of the other solutions that extend the Maximin, the Leximin, the Kalai-Smorodinsky, and the lexicographic Kalai-Smorodinsky solutions of convex bargaining problems are quite different: they all are closely connected with various versions of expansion consistency and domination.

In our studies of the Utilitarian solution and the two new solutions combining the Maximin solution and the Utilitarian solution in different ways: Maximin-Utilitarian solution and Utilitarian-Maximin solution, we note that it is probably the first time that the axioms relating to the Pigou-Dalton domination are used in the characterizations of the solutions in bargaining theory.

Finally we discuss the egalitarian solution and present some approaches to study the egalitarian solution for finite problems. Depending on choices of the distance function and the “reference point” from which the distance is calculated between a point in the solution and this reference point, we can have various egalitarian solutions.

Our axiomatic characterizations of the extensions of the various standard solutions of convex bargaining problems to finite bargaining problems have shed new light on the behavior of these solutions. The difference between Utilitarian-Maximin solution and Maximin-Utilitarian solution and the complexity of the Egalitarian solution arising from finite bargaining problems are evidences that finite problems are unique and are more complicated than bargaining problems with comprehensive feasible sets. As a future research agenda, it would be interesting to explore other possibilities.

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