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Generic Continuous Functions and other Strange Functions in Classical Real Analysis

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GENERIC CONTINUOUS FUNCTIONS AND OTHER STRANGE FUNCTIONS

IN CLASSICAL REAL ANALYSIS

by

Douglas A. Woolley

Under the direction of Mihaly Bakonyi

ABSTRACT

In this paper we examine continuous functions which on the surface seem to defy well-known mathematical principles. Before describing these functions, we introduce the Baire Category theorem and the Cantor set, which are critical in describing some of the functions and counterexamples. We then describe generic continuous functions, which are nowhere differentiable and monotone on no interval, and we include an example of such a function. We then construct a more conceptually challenging function, one which is everywhere differentiable but monotone on no interval. We also examine the Cantor function, a nonconstant continuous function with a zero derivative almost everywhere. The final section deals with products of derivatives.

INDEX WORDS: Baire, Cantor, Generic Functions, Nowhere Differentiable

GENERIC CONTINUOUS FUNCTIONS AND OTHER STRANGE FUNCTIONS IN CLASSICAL REAL ANALYSIS

by

Douglas A. Woolley

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

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in the College of Arts and Sciences

Georgia State University

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TABLE OF CONTENTS

List of Figures v

Section

1 Introduction 1

2 The Cantor Set 2

3 Baire's Theorem 4

4 Generic Continuous Functions 6

5 An Everywhere Differentiable,Nowhere Monotone Function 13

6 The Cantor Function 20

7 Products of Derivatives 23

References 27

List of Figures

Figure 6.1 20

1 Introduction

Most continuous functions used in applications have several additional properties, such as differentiability at most points of the domain, or being monotone on certain intervals. It turns out, however, that the *generic* continuous function (a notion defined in Section 2) is not differentiable at any point, nor is it monotone on any subinterval of the domain. This is rather unexpected, and shows that the generic continuous function is quite different from the ones used in most applications. In this thesis we investigate this and other strange behaviors of continuous real-valued functions. We construct an everywhere differentiable, nowhere monotone function. Also presented is the Cantor function (or "Devil's Staircase"), which is continuous, has derivative zero almost everywhere, but is not constant. Finally, we deal with functions which are derivatives of other functions. We first present an example of two such functions the product of which is not a derivative. A proof is then presented that the product of a derivative and an absolutely continuous function is always a derivative.

The material in this thesis is organized as follows: Section 2 describes the Cantor set, which is of great importance in Real Analysis and plays a key role in constructing several counterexamples. Section 3 is a presentation of Baire's Theorem, which is critical to Section 4, which shows that the generic continuous function is nowhere differentiable and is monotone on no subinterval. An example of such a function is then constructed. Section 5 describes a function which is everywhere differentiable but monotone on no subinterval. In Section 6 we present the Cantor function, while Section 7 deals with products of derivatives.

2 The Cantor Set

In the study of Real Analysis, it is critical to understand the unusual properties of the Cantor set. Many examples and still more counterexamples make use of this set. Its construction appears in numerous texts; see for example [1], [6] and [8].

Consider the closed interval [0,1]. Remove the open middle third $(\frac{1}{3}, \frac{2}{3})$. Then remove the open middle thirds from the two remaining segments. At this point the segments $[1, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$ remain. Continue removing the open middle thirds of the remaining segments. After infinitely many steps what remains is a subset of the real numbers known as the Cantor set C . Since the Cantor set results from removing only open intervals, the complement of the Cantor set is open, therefore the Cantor set is closed. The measure of the complement is the sum of the measures of the removed intervals,

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{1}{3}\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \frac{1}{3} \cdot 3 = 1.$$

Therefore, the Cantor set has measure zero. Despite this, the Cantor set is uncountable. To prove this, we consider the ternary representations of the numbers in the unit interval. Since, in base three, $\frac{1}{3}$ is equal to 0.1, and $\frac{2}{3}$ is equal to 0.2, it is evident that, in removing the middle third of the unit interval, we actually removed all of the real numbers whose ternary decimal representation have a 1 in the first decimal place, except for 0.1 itself. We can, however, choose to represent $\frac{1}{3}$ as $0.0\overline{2}$. No other number in the interval can be represented in ternary decimal representation without a 1 in the first decimal place. Similarly, when removing the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, we remove all remaining real numbers whose

ternary representations have a 1 in the second decimal place, again with the exceptions of the leftmost endpoints, $\frac{1}{9}$ and $\frac{7}{9}$. As with $\frac{1}{3}$, these can be represented in ternary either with or without using a 1; $\frac{1}{9}$ as 0.01 or $0.00\overline{2}$, and $\frac{7}{9}$ as 0.21 or $0.20\overline{2}$. Continuing, each nth step of the process removes all real numbers in the interval that cannot be expressed in ternary notation without a 1 in the nth decimal place. Therefore, the Cantor set consists of all real numbers in the interval $[0, 1]$ that have a ternary decimal representation consisting of only 0's and 2's. There exists a surjective function from the Cantor set to the interval $[0, 1]$. It is defined by mapping each ternary number in the Cantor set to the binary number which results from replacing each 2 in the ternary number with a 1. Since the Cantor set consists of all ternary numbers consisting of only 2's and 0's after the decimal point, the range of this function then consists of all binary numbers in the interval. While this function is clearly surjective, it is not one-to-one; the two endpoints of each removed interval map to the same binary number. Since it is a surjective function, card $C \geq$ card $[0, 1]$. Since $C \subseteq [0, 1]$, card $C \leq$ card $[0, 1]$. Hence, card $C =$ card $[0, 1]$, and therefore C is uncountable. It is easily shown that the Cantor set is nowhere dense. Since C is generated by removing only open sets from the unit interval, its complement is the union of open sets and is therefore an open set. Hence, C is closed and so $\overline{C} = C$. Since C is a zero set, it contains no interval and therefore no interior points. Therefore the interior of $\overline{C}$ is empty, so C is nowhere dense.

3 Baire's Theorem

Baire's Theorem is a fundamental result in the study of Real Analysis. Its proof appears in most textbooks on the subject. Since it is critical to Section 4, its proof is included as presented in [5] and [6].

Definition Let M be a metric space. A countable intersection $G = \bigcap_{n=1}^{\infty} G_n$ of open dense subsets of M is called a *thick* subset of M. The complement of a thick set is *thin* (or *meager*).

The empty set is always thin and the full space M is always thick in itself. A single open dense subset is thick and a single nowhere dense subset is thin. The Cantor set is thin in [0,1], and $\mathbb{R}$ is thin in $\mathbb{R}^2$.

Theorem 3.1. (Baire's Theorem) *Every thick subset of a complete metric space M is dense in M. A nonempty, complete metric space is not thin; if M is the countable union of closed sets, then at least one has a nonempty interior.*

Proof. Let M be a complete metric space, $M \neq \emptyset$. Let $G = \bigcap_{n=1}^{\infty} G_n$ be a thick subset of M, with each G_n open and dense in M. To prove that G is dense in M, it suffices to show that $G \cap A \neq \emptyset$ for each nonempty open set A of M. Suppose there exists an open, nonempty set A such that $G \cap A = \emptyset$. Therefore, $A \subset G^c$ Thus,

$$G^c \cap A = A,$$

$$(\bigcap_{n=1}^{\infty} G_n)^c \cap A = A,$$

$$\bigcup_{n=1}^{\infty} (G_n^c \cap A) = A,$$

and therefore A is thin. Since no nonempty open subset of a complete metric space can be thin, A must be empty, a contradiction. Therefore, G is dense in M.

To prove that a complete metric space M is not thin, assume the converse. Suppose that $M = \bigcup_{n=1}^{\infty} K_n$, and each K_n has an empty interior. Then, each $G_n = K_n^c$ is open and dense, therefore,

$$G = \bigcap_{n=1}^{\infty} G_n = (\bigcup_{n=1}^{\infty} K_n)^c = \emptyset,$$

contradicting the density of G. $\square$

4 Generic Continuous Functions

In this section we will prove a quite surprising result, namely that the generic function in $C^0([a,b],\mathbb{R})$ is nowhere differentiable and monotone on no interval. Its presentation is based on [6]. We later construct an example of such a function (based on results in [3], [6], and [7]).

Definition. If all points in a thick subset of M have some property, then that property is said to be a *generic* property of M. This is the same as saying that most points of M possess that property.

For example, the generic point of [0,1] is not in the Cantor set, and the generic point of $\mathbb{R}^2$ does not lie on the x-axis. The set $\mathbb{Q}$ is thin in $\mathbb{R}$; it is a countable union of points, and each point is a closed, nowhere dense set. Hence, the generic real number is irrational.

Definition. We denote by $C^0 = C^0([a,b],\mathbb{R})$ the set of all continuous, real-valued functions defined on the closed interval $[a,b]$. For $f \in C^0([a,b],\mathbb{R})$, let $\|f\| = \max\{|f(x)| : x \in [a,b]\}$.

Theorem 4.1. *The generic function $f \in C^0([a,b],\mathbb{R})$ is nowhere differentiable on $[a,b]$, and is not monotone on any subinterval of $[a,b]$.*

Definition. If $\phi : [a,b] \to \mathbb{R}$ is continuous and its graph consists of finitely many line segments in $\mathbb{R}^2$, then ϕ is called a *piecewise linear* function.

Before proving Theorem 4.1, it is necessary to prove two lemmas.

Lemma 4.2. *The set PL of piecewise linear functions is dense in C^0.*

Proof. Let $f \in C^0$ and $\epsilon > 0$ be given. Since $[a,b]$ is compact, f is uniformly continuous.

Therefore, there exists a $\delta > 0$ such that $|t - s| < \delta$ implies $|f(t) - f(s)| < \epsilon$. Choose $n > (b - a)/\delta$ and partition $[a, b]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$, each interval of measure $< \delta$. Let $\phi : [a, b] \to \mathbb{R}$ be the piecewise linear function consisting of the segments joining the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ on the graph of f. For $t \in I_i$, $\phi(t)$ is between $f(x_{i-1})$ and $f(x_i)$. Both $f(x_{i-1})$ and $f(x_i)$ differ from $f(t)$ by less than ϵ. Therefore, for all $t \in [a, b]$,

$$|f(t) - \phi(t)| < \epsilon.$$

Therefore, every continuous function is the limit of a sequence of piecewise linear functions, and so PL is dense in C^0. $\square$

Definition A continuous periodic function whose graph consists of line segments of equal length and slopes $\pm\alpha$ for some real number α is called a *sawtooth* function.

Lemma 4.3. *If $\phi \in PL$ and $\epsilon > 0$ are given, then there exists a sawtooth function σ such that $\| \sigma \| \leq \epsilon$, σ has period $\leq \epsilon$, and*

$$\min\{|\text{slope}\sigma|\} > \max\{|\text{slope}\phi|\} + 1/\epsilon.$$

Proof. Define σ_0 as follows:

$$\sigma_0(x) = |x| \qquad \text{if } |x| \leq 1 \tag{1}$$

$$\sigma_0(x + 2p) = \sigma_0(x) \text{ if } x \in \mathbb{R} \text{ and } p \in \mathbb{Z}. \tag{2}$$

Let $\theta = \max\{|\text{slope}\phi|\}$ and choose c large. The compressed sawtooth function $\sigma(x) = \epsilon\sigma_0(cx)$ has $\|\sigma\| = \epsilon$, period $T = 1/c$, and slope $m = \pm\epsilon c$. When c is sufficiently large, $T \leq \epsilon$, and $|m| > \theta + 1/\epsilon$. $\square$

Proof of Theorem 4.1.

For $n \in \mathbb{N}$, define:

$$R_n = \{f \in C^0 : \forall x \in [a, b - \frac{1}{n}] \ \exists h > 0 \text{ such that } |\frac{\Delta f}{h}| > n\},$$

$$L_n = \{f \in C^0 : \forall x \in [a + \frac{1}{n}, b] \ \exists h < 0 \text{ such that } |\frac{\Delta f}{h}| > n\},$$

and

$$G_n = \{f \in C^0 : f \text{ restricted to any interval of length } \frac{1}{n} \text{ is non-monotone}\},$$

where $\Delta f = f(x + h) - f(x)$. We prove that each of these sets is open and dense in C^0.

To show that these sets are dense in C^0, it suffices to prove that the closure of each set contains PL, since by Lemma 4.2 the closure of PL is C^0. Let $\phi \in PL$ and $\epsilon > 0$ be given. By Lemma 4.3, there is a sawtooth function σ such that $\|\sigma\| \leq \epsilon$, σ has period $T < 1/n$, and

$$\min\{|\text{slope}\sigma|\} > \max\{|\text{slope}\phi|\} + n.$$

The slopes of the piecewise linear function $f = \phi + \sigma$ are dominated by those of σ, and therefore alternate in sign with period $< 1/2n$. At any $x \in [a, b - \frac{1}{n}]$, there is a slope either

$> n$ or $< -n$, therefore $f \in R_n$. Similarly, for $x \in [a + \frac{1}{n}, b]$, there is also a slope either $> n$ or $< -n$, therefore $f \in L_n$. Also, any interval of measure $1/n$ contains in its interior either a maximum or a minimum of σ, and therefore contains a strictly increasing subinterval as well as a strictly decreasing subinterval. Therefore, $f \in G_n$. Since f is arbitrarily close to ϕ, and ϕ is arbitrarily close to $g \in C^0$, it follows that f is arbitrarily close to g, and therefore R_n, L_n, and G_n are dense in C^0. Next, we show that these are open sets. Let $f \in R_n$ be given. For each $x \in [a, b - \frac{1}{n}]$, there exists an $h = h(x) > 0$ such that

$$\left|\frac{f(x+h) - f(x)}{h}\right| > n.$$

Since f is continuous, there exists a neighborhood T_x of x, $T_x \subset [a, b]$, and a constant $\nu > 0$, such that for the same h,

$$\left|\frac{f(t+h) - f(t)}{h}\right| > n + \nu$$

for all $t \in T_x$. Since $[a, b - \frac{1}{n}]$ is compact, finitely many such neighborhoods cover it. Let $T_{x_1}, ..., T_{x_m}$ be a finite set of neighborhoods which covers $[a, b - \frac{1}{n}]$. Then f is continuous, therefore for all $t \in \overline{T}_{x_i}$,

$$\left|\frac{f(t+h_i) - f(t)}{h_i}\right| \geq n + \nu_i,$$

where $h_i = h(x_i)$ and $\nu_i = \nu(x_i)$. These m inequalities for points t in the m sets T_{x_i} remain valid if f is replaced by a continuous function g with $\|f - g\|$ sufficiently small. Therefore,

$$\left|\frac{g(t+h_i) - g(t)}{h_i}\right| > n,$$

which implies that $g \in R_n$, and hence f is in the interior of R_n. Therefore, R_n is open in C^0. A similar proof shows that L_n is also open in C^0. To show that G_n is open, let (f_k) be a sequence of functions in G_n^c and f_k converges uniformly to f. Since $f_k \notin G_n$, each f_k is monotone on some interval $I_k = [s_k, t_k]$ of measure $1/n$. There are subsequences of these endpoints which converge to limits s and t. Since the intervals are all of measure $\frac{1}{n}$, the interval $[s, t]$ is also of measure $\frac{1}{n}$. By uniform convergence, f is monotone on I. Therefore, G_n^c is closed, and so G_n is open. Since R_n, L_n, and G_n are open and dense in C^0, the set

$$\bigcap_{n=1}^{\infty} R_n \cap L_n \cap G_n$$

is a thick set. Let f be a function belonging to this set. For each $x \in [a, b]$, there is a sequence $h_n \neq 0$, such that

$$\left|\frac{f(x+h_n) - f(x)}{h_n}\right| > n.$$

The numerator of this quotient is at most $2\|f\|$, so $h_n \to 0$ as $n \to \infty$. Therefore, f is not differentiable at x. Also, f is non-monotone on every interval of measure $1/n$. Any interval, no matter how small, contains an interval of measure $\frac{1}{n}$ when n is sufficiently large. Therefore, f is not monotone on any interval.

Example. ([3],[6], and [7]) We will now construct a continuous function that is nowhere differentiable and monotone on no interval.

Define $\sigma_0 : \mathbb{R} \to \mathbb{R}$ as in (1) and (2). Let $\sigma_k(x) = (\frac{3}{4})^k \sigma_0(4^k x)$, for all $k \in \mathbb{Z}$. Clearly, each σ_k is continuous on $\mathbb{R}$, has period $T_k = \frac{2}{4^k}$, amplitude $A_k = (\frac{3}{4})^k$, and derivative $\sigma_k' = \pm 3^k$. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \sum_{k=0}^{\infty} \sigma_0(x) \tag{3}$$

Since the amplitudes form a converging series, $\{\sigma_k\}$ converges uniformly according to the Weierstrass M-test, and therefore f is continuous on $\mathbb{R}$ since uniform convergence preserves continuity. To show that f fails to have a finite derivative at any point, fix an arbitrary $x \in \mathbb{R}$. Let $\delta_n = 1/2 \cdot 4^n$. Clearly, $\delta_n \to 0$ as $n \to \infty$.

$$\frac{\Delta f}{\Delta x} = \frac{f(x \pm \delta_n) - f(x)}{\delta_n} = \sum_{k=0}^{\infty} \frac{\sigma_k(x \pm \delta_n) - \sigma_k(x)}{\delta_n}. \tag{4}$$

Whenever $k > n$,

$$\begin{aligned} \sigma_k(x \pm \delta_n) - \sigma_k(x) &= (\frac{3}{4})^k [\sigma_0(4^k(x \pm \delta_n)) - \sigma_0(4^k x)] \\ &= (\frac{3}{4})^k [\sigma_0(4^k x \pm \frac{4^{k-n}}{2}) - \sigma_0(4^k x)], \end{aligned}$$

which equals zero because σ_0 has period T = 2. Therefore, (4) becomes

$$\frac{\Delta f}{\Delta x} = \sum_{k=0}^{n-1} \frac{\sigma_k(x \pm \delta_k) - \sigma_k(x)}{\delta_k} + \frac{\sigma_n(x \pm \delta_k) - \sigma_n(x)}{\delta_n}.$$

The slope of the last term of this sum is $\pm 3^n$, while the maximum absolute value of the sum of the slopes of the first n terms is $1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{3-1} < \frac{3^n}{2}$. Since $3^n \to \infty$ as $n \to \infty$, f fails to have a finite derivative at any point. Also, since the slope of each sum is dominated by the nth term, the intervals of monotonicity of each sum partial sum $\sum_{k=0}^{n} \sigma_k(x)$ will be the intervals of monotonicity of σ_n. Each σ_n has period $\mathrm{T}_n = 2/4^n$, and therefore

is monotone on intervals of measure $\frac{1}{4^n}$, which go to zero as $n \to \infty$. Therefore f is not monotone on any interval.

5 An Everywhere Differentiable, Nowhere Monotone Function

The purpose of this section is to present an example of an everywhere differentiable, nowhere monotone function. Examples of such functions were given by Köpcke in 1889 and Pereno in 1897. The much simpler construction presented here was developed by Katznelson and Stromberg [4]. For its presentation we begin with a series of lemmas.

Lemma 5.1. *Let $r, s \in \mathbb{R}$. Then:*

(i) If $r > s > 0$, then

$$\frac{r-s}{r^2-s^2} < \frac{2}{r}$$

and

(ii) If $r > 1$ and $s > 1$, then

$$\frac{r+s-2}{r^2+s^s-2} < \frac{2}{s}.$$

Proof. Assertion (i) is obvious. Proof of assertion (ii).

If $r > 1$ and $s > 1$, then

$$5 < (r-s)^2 + (r-1)(s-1) + r^2 + 3s + r$$

$$5 < (r^2 - 2rs + s^2) + (rs - r - s + 1) + r^2 + 3s + r$$

$$5 < 2r^2 + s^2 - rs + 2s + 1$$

$$rs + s^2 - 2s < 2r^2 + 2s^2 - 4$$

$$s(r+s-2) < 2(r^2+s^2-2)$$

$$\frac{r+s-2}{r^2+s^2-2} < \frac{2}{s}.$$

□

Lemma 5.2. *Let $\phi(x) = (1+|x|)^{-\frac{1}{2}}$ for $x \in \mathbb{R}$. Then*

$$\frac{1}{b-a}\int_a^b \phi(x)\,dx < 4\min\{\phi(a), \phi(b)\}$$

whenever a and b are distinct real numbers.

Proof. Without loss of generality, assume that $a < b$. If $0 \leq a$, then Lemma 5.1 verifies that

$$\begin{aligned}\frac{1}{b-a}\int_a^b \phi(x)\,dx &= \frac{2(\sqrt{1+b}-\sqrt{1+a})}{(1+b)-(1+a)} \\ &< \frac{4}{\sqrt{1+b}} \ = 4\min\{\phi(a), \phi(b)\}.\end{aligned}$$

Since ϕ is an even function, the case that $b \leq 0$ is also verified by the above proof. The case that $a < 0 < b$ follows from the second part of Lemma 5.1:

$$\begin{aligned}\frac{1}{b-a}\int_a^b \phi(x)\,dx &= \frac{2(\sqrt{1+b}-\sqrt{1-a}-2)}{(1+b)-(1-a)-2} \\ &< 4\min\{\phi(a), \phi(b)\}\end{aligned}$$

. □

Lemma 5.3. *If ϕ is as in Lemma 5.2, and ψ is any function of the form*

$$\psi(x) = \sum_{j=1}^{n} c_j \phi(\lambda_j(x-\alpha_j)),$$

where $c_1, ..., c_n$ and $\lambda_1, ..., \lambda_n$ are positive real numbers, and $\alpha_1, ..., \alpha_n$ are any real numbers, then

$$\frac{1}{b-a}\int_a^b \psi(x)\,dx < 4\min\{\psi(a), \psi(b)\},$$

whenever a and b are distinct real numbers.

Proof. The result follows from Lemma 5.2 and the fact that

$$\frac{1}{b-a}\int_a^b \phi(\lambda(x-\alpha))\,dx = \frac{1}{\lambda(b-\alpha)-\lambda(a-\alpha)}\int_{\lambda(a-\alpha)}^{\lambda(b-\alpha)} \phi(t)\,dt.$$

□

Lemma 5.4. *Let $(\psi_n)_{n=1}^{\infty}$ be any sequence of functions as in Lemma 5.3. For $x \in \mathbb{R}$ and each n define*

$$\Psi_n(x) = \int_0^x \psi_n(t)\,dt.$$

Suppose that $\sum_{n=1}^{\infty} \psi_n(a) = s < \infty$ for some $a \in \mathbb{R}$. Then the series $F(x) = \sum_{n=1}^{\infty} \Psi_n(x)$ converges uniformly on every bounded subset of $\mathbb{R}$, the function F is differentiable at a, and $F'(a) = s$. In particular, if $\sum_{n=1}^{\infty} \psi_n(t) = f(t) < \infty$ for all $t \in \mathbb{R}$, then F is differentiable everywhere on $\mathbb{R}$ and $F' = f$.

Proof. Let b satisfy $b \geq |a|$. Then, using Lemma 5.3, $-b \leq x \leq b$ implies

$$|\Psi_n(x)| \leq \left|\int_0^a \psi_n(t)dt\right| + \left|\int_a^x \psi_n(t)dt\right|$$

$$\leq 4|a|\psi_n(a) + 4|x-a|\psi_n(a) \leq 12b\psi_n(a).$$

Uniform convergence on $[-b, b]$ follows from the Weierstrass M-test. To prove that $F'(a) = s$, let $\epsilon > 0$ be given. Choose N such that

$$10 \cdot \sum_{n=N+1}^{\infty} \psi_n(a) < \epsilon.$$

Since each ψ_n is continuous at a, there exists some $\delta > 0$ such that

$$\left| \frac{1}{h} \int_a^{a+h} \psi_n(t)\, dt - \psi_n(a) \right| < \frac{\epsilon}{2N},$$

whenever $0 < |h| < \delta$ and $1 \leq n \leq N$. Therefore, using Lemma 5.3, $0 < |h| < \delta$ implies that

$$\begin{aligned}
\left| \frac{F(a+h) - F(a)}{h} - s \right| &= \left| \sum_{n=1}^{\infty} \left\{ \frac{1}{h} \int_a^{a+h} \psi_n(t) dt - \psi_n(a) \right\} \right| \\
&\leq \sum_{n=1}^{N} \left| \frac{1}{h} \int_a^{a+h} \psi_n(t) dt - \psi_n(a) \right| \\
&+ \sum_{n=N+1}^{\infty} \left\{ \frac{1}{h} \int_a^{a+h} \psi_n(t) dt + \psi_n(a) \right\} \\
&< \frac{\epsilon}{2} + \sum_{n=N+1}^{\infty} 5\psi_n(a) < \epsilon.
\end{aligned}$$

□

Lemma 5.5. *Let $I_1, ..., I_n$ be disjoint open intervals, let α_j be the midpoint of I_j, and let ϵ and $y_1, ..., y_n$ be positive real numbers. Then there exists a function ψ as in Lemma 5.3 such that for each j,*

(i) $\psi(\alpha_j) > y_j$,

(ii) $\psi(x) < y_j + \epsilon$ if $x \in I_j$,

(iii) $\psi(x) < \epsilon$ if $x \notin I_1 \cup \cdots \cup I_n$.

Proof. Choose $c_j = y_j + \frac{\epsilon}{2}$ and write $\phi_j(x) = c_j\phi(\lambda_j(x - \alpha_j))$, where λ_j is chosen sufficiently large that $\phi_j(x) < \frac{\epsilon}{2}$ if $x \notin I_j$. Take $\psi = \phi_1 + \cdots + \phi_n$. Since $I_j \cap I_k = \emptyset$ when $j \neq k$, and since ϕ_j takes its maximum value at α_j, properties (i), (ii), and (iii) are satisfied. $\square$

Theorem 5.6. *Let $\{\alpha_j\}_{j=1}^{\infty}$, and $\{\beta_j\}_{j=1}^{\infty}$ be disjoint countable subsets of $\mathbb{R}$. Then there exists a real-valued, everywhere differentiable function F on $\mathbb{R}$ satisfying $F'(\alpha_j) = 1$, $F'(\beta_j) < 1$ for all j, and $0 < F'(x) \leq 1$ for all x.*

Proof. We obtain F as in Lemma 5.4 by first constructing $F' = f = \sum_{n=1}^{\infty} \psi_n$, with partial sums $f_n = \sum_{k=1}^{n} \psi_k$, in such a way that

$$\begin{aligned} A_n &: f_n(\alpha_j) > 1 - \frac{1}{n} \qquad (1 \leq j \leq n), \\ B_n &: f_n(x) < 1 - \frac{1}{n+1} \qquad x \in \mathbb{R} \\ C_n &: \psi_n(\beta_j) < \frac{1}{2n \cdot 2^n} \qquad (1 \leq j \leq n). \end{aligned}$$

Supposing that this were done we would have

$$F'(\alpha_j) = \lim_{n\to\infty} f_n(\alpha_j) = 1,$$

$$0 < F'(x) = \lim_{n\to\infty} f_n(x) \leq 1,$$

and, choosing $n > j$,

$$F'(\beta_j) = f_{n-1}(\beta_j) + \sum_{k=n}^{\infty} \psi_k(\beta_j)$$

$$< 1 - \frac{1}{n} + \sum_{k=n}^{\infty} \frac{1}{2k \cdot 2^k} < 1 - \frac{1}{n} + \frac{1}{2n} \cdot 1 = 1 - \frac{1}{2n} < 1,$$

and thus we would have the desired F. Proceeding inductively, choose an open interval I with midpoint α_1 such that $\beta_1 \notin I$. Then apply Lemma 5.5 with $\epsilon = y_1 = \frac{1}{4}$ to obtain $f_1 = \psi_1$ that satisfies A_1, B_1, and C_1. Now suppose that $n > 1$ and f_{n-1} and ψ_{n-1} have been chosen which satisfy A_{n-1}, B_{n-1}, and C_{n-1}. Select disjoint open intervals $I_1, ..., I_n$ such that, for each $j \in \{1, ..., n\}, \alpha_j$ is the midpoint of $I_j, I_j \cap \{\beta_1, ..., \beta_n\} = \emptyset$, and $f_{n-1}(x) < f_{n-1}(\alpha_j)$, where

$$\delta = \frac{1}{n(n+1)} - \frac{1}{2n \cdot 2^n} > 0.$$

Now, apply Lemma 5.5, with $\epsilon = \frac{1}{2n \cdot 2^n}$ and $y_j = 1 - \frac{1}{n} - f_{n-1}(\alpha_j)$, with $1 \leq j \leq n$, to obtain ψ_n. C_n is now satisfied. Also

$$f_n(\alpha_j) = f_{n-1}(\alpha_j) + \psi_n(\alpha_j) > f_{n-1}(\alpha_j) + y_j = 1 - \frac{1}{n},$$

and so A_n obtains. To check B_n, we note that if $x \in I_j$, then

$$f_n(x) = f_{n-1}(x) + \psi_n(x)$$

$$< f_{n-1}(\alpha_j) + \delta + y_j + \epsilon$$

$$= 1 - \frac{1}{n} + \frac{1}{n(n+1))} = 1 - \frac{1}{n+1};$$

while if $x \notin \bigcup_{j=1}^{n} I_j$, then

$$f_n(x) = f_{n-1}(x) + \psi_n(x) < 1 - \frac{1}{n} + \epsilon < 1 - \frac{1}{n-1}.$$

□

Corollary 5.7. *There exists a real-valued, everywhere differentiable function H on $\mathbb{R}$ such that H is monotone on no subinterval of $\mathbb{R}$ and H' is bounded.*

Proof. Let $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ be disjoint dense subsets of $\mathbb{R}$. Apply he preceding theorem to obtain everywhere differentiable functions F and G on $\mathbb{R}$ such that

$$F'(\alpha_j) = G'(\beta_j) = 1, \quad G'(\alpha_j) < 1, \;\; F'(\beta_j) < 1,$$

$$0 < F'(x) \leq 1, \qquad 0 < G'(x) \leq 1,$$

for all j and x. Now write $H = F - G$. Then

$$H'(\alpha_j) > 0, \; H'(\beta_j) < 0, \text{ and } -1 < \mathrm{H'(x)} < 1,$$

for all j and x. Since $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ are both dense, H cannot be monotone on any interval. $\square$

6 The Cantor Fuction

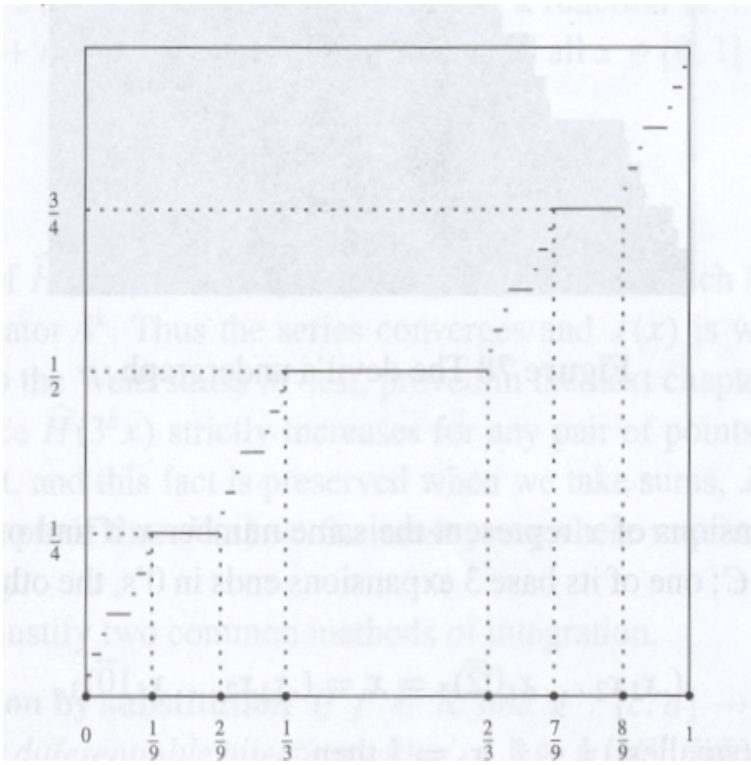

Figure 6.1

The Cantor function ([1] and [6]), $H : [0,1] \to [0,1]$ is a continuous function whose derivative is zero almost everywhere, but is not constant. It is sometimes referred to as the "Devil's Staircase" function, because it has infinitely many steps. It is constant on the closure of each discarded interval of the Cantor set. It is defined as

$$H(x) = \begin{cases} \frac{1}{2} & \text{if} \qquad x \in [\frac{1}{3}, \frac{2}{3}] \\ \frac{1}{4} & \text{if} \qquad x \in [\frac{1}{9}, \frac{2}{9}] \\ \frac{3}{4} & \text{if} \qquad x \in [\frac{7}{9}, \frac{8}{9}] \\ \frac{1}{8} & \text{if} \qquad x \in [\frac{1}{27}, \frac{2}{27}] \\ \dots & \dots \end{cases}$$

Because H is constant on each discarded interval of the Cantor set, it is clearly differentiable on the complement of the Cantor set. Since the Cantor set has measure zero, H is differentiable almost everywhere. To show that H is continuous, we once again use binary and ternary representations. If $x \in [0, 1]$ has ternary representation $x = 0.x_1x_2x_3...$, then $y = H(x)$ has binary representation $y = 0.y_1y_2y_3...$, where

$$y_i = \begin{cases} 0 & \text{if} \qquad \exists\, k < i \text{ such that } x_k = 1 \\ 1 & \text{if} \qquad x_i = 1 \text{ and } \nexists\, k < i \text{ such that } x_k = 1 \\ \frac{x_i}{2} & \text{if} \quad x_i = 0 \text{ or } x_i = 2 \text{ and } \nexists\, k < i \text{ such that } x_k = 1 \end{cases}$$

H is obviously continuous on its constant intervals, so it remains to show that it is continuous at each point in the Cantor set. Let x be an element of the Cantor set. Let $\epsilon = 3^{-n}$. Any element z of the Cantor set in the ϵ neighborhood of x differs from x by a number whose ternary expansion begins with at least n zeros. Therefore, $f(z)$ differs from $f(x)$ by at most 2^{-n}. If a point belongs to the complement of the Cantor set, it lies on one of the constant intervals, and therefore its y-value is the same as the y-value of the endpoints of the interval.

Since one of the endpoints must also be in the same ϵ neighborhood, the y-value must differ from $f(x)$ by at most 2^{-n}, and therefore H is continuous.

7 Products of Derivatives

It is well known that every continuous, real-valued function is a derivative. It is obvious that the sum or difference of any two derivatives is also a derivative. Later in this section we will prove that the product of a derivative and an absolutely continuous function is a derivative. However, the product of two derivatives is in general not a derivative. We now present a counterexample, originating in [9] (see also [2]), to an even stronger situation, when the product of a continuous function and a derivative fails to be a derivative. Let

$$f(x) = \begin{cases} x^{\frac{1}{2}} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

$$g(x) = \begin{cases} x^{\frac{1}{2}} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

and

$$h(x) = \begin{cases} x^{\frac{3}{2}} \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Since $0 \leq |f(x)| \leq x^{\frac{1}{2}}$, the pinching theorem shows that f is continuous at zero, and therefore continuous on $\mathbb{R}$. The same is true of h. Now, $h'(x) = \frac{3}{2}x^{\frac{1}{2}} \cos \frac{1}{x} + x^{-\frac{1}{2}} \sin \frac{1}{x}$ when $x \neq 0$. Hence, $h'(x) - g(x)$ is continuous and $g(x)$ is therefore a derivative. To show that $fg(x)$ is not a derivative, it is necessary to first show that the function

$$\alpha(x) = \begin{cases} \cos\frac{2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a derivative.

Consider the function

$$\gamma(x) = \begin{cases} \frac{x^2}{2}\sin\frac{2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for which we have

$$\gamma'(x) = \begin{cases} x\sin\frac{2}{x} - \cos\frac{2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Once again, the pinching theorem shows that the first term is continuous, and therefore $\alpha(x)$ is also a derivative. Now,

$$fg(x) = \begin{cases} \sin^2(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This can be rewritten, using the double angle identity, as

$$fg(x) = \begin{cases} \frac{1}{2}(1 - \cos\frac{2}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Adding $2fg + \cos\frac{2}{x}$ gives the function

$$\begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Clearly, this is not a derivative. Therefore, fg cannot be a derivative.

Definition A function $F : [a, b] \to \mathbb{R}$ is called *absolutely continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that, if $n \in \mathbb{N}, a \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq ... \leq a_n \leq b_n \leq b$ and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$. F' exists almost everywhere on $[a, b]$ and is Lebesgue integrable on $[a, b]$. For all $x, y \in [a, b]$, with $a \leq x \leq y \leq b$,

$$F(y) - F(x) = \int_x^y F'(t)\, dt.$$

Theorem 7.1. ([2]) *The product of a derivative and an absolutely continuous function is a derivative.*

Proof. Let $F : [0, 1] \to \mathbb{R}$ be absolutely continuous,with $f(x) = F'(x)$ almost everywhere, and let $G : [0, 1] \to \mathbb{R}$ be differentiable with $G'(x) = g(x)$, and $G(a) = 1$ for some $a \in [0, 1]$. To show that $F \cdot g(x)$ is the derivative of

$$F(x)G(x) - \int_0^x f(t)G(t)\, dt,$$

it suffices to show that

$$\lim_{x \to a} \frac{1}{x - a}(F(x)G(x) - F(a)G(a) - \int_a^x f(t)G(t)\, dt) - F(a)g(a)$$

exists and is equal to 0 at everywhere on [0,1]. This limit is equivalent to

$$\lim_{x \to a} \frac{1}{x - a}(F(x)G(x) - F(x)G(a) - F(a)g(a) + F(x) - F(a)G(a) - \int_a^x f(t)G(t)\, dt).$$

Since F is continuous, and $G'(a) = g(a)$, this becomes

$$\lim_{x\to a}\frac{1}{x-a}((F(x)-F(a))G(a)-\int_a^x f(t)G(t)\,dt).$$

Since F is absolutely continuous,

$$F(x)-F(a)=\int_x^a f(t)\,dt.$$

Therefore, the limit, provided it exists, is equal to

$$\lim_{x\to a}\frac{1}{x-a}\int_a^x (f(t)G(t)-f(t)G(a))\,dt.$$

Since $G'(a)=g(a)$, it follows that $G(t)-G(a)=(g(a)+\epsilon(t))(t-a)$, where $\epsilon(t)\to 0$ as $t\to a$. Hence,

$$\begin{aligned}\left|\frac{1}{x-a}\int_a^x (f(t)G(t)-f(t)G(a))\,dt\right| &= \left|\frac{1}{x-a}\int_a^x (f(t)(g(a)+\epsilon(t))(t-a))\,dt\right| \\ &\le \frac{|g(a)|+|\epsilon(x)|}{|x-a|}\int_a^x |f(t)(t-a)|dt \\ &\le (|g(a)|+|\epsilon(x)|)\int_a^x |f(t)|\,dt.\end{aligned}$$

$f(t)$ is Lebesgue integrable, and therefore $|f(t)|$ is Lebesgue integrable as well, and so

$$\int_a^x |f(t)|\,dt\to 0 \text{ as } x\to a.$$

Therefore the original limit exists and equals zero. □

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