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ACCEPTANCE

This dissertation, A COGNITIVE MODEL OF ALGEBRA ACHIEVEMENT AMONG UNDERGRADUATE COLLEGE STUDENTS, by TAMMY DAUN TOLAR, was prepared under the direction of the candidate's Dissertation Advisory Committee. It is accepted by the committee members in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Education, Georgia State University.

The Dissertation Advisory Committee and the student's Department Chair, as representatives of the faculty, certify that this dissertation has met all standards of excellence and scholarship as determined by the faculty. The Dean of the College of Education concurs.

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ABSTRACT

A COGNITIVE MODEL OF ALGEBRA ACHIEVEMENT AMONG UNDERGRADUATE COLLEGE STUDENTS
by
Tammy D. Tolar

Algebra has been called a gatekeeper because proficiency in algebra allows access to educational and economic opportunities. Many students struggle with algebra because it is cognitively demanding. There is little empirical evidence concerning which cognitive factors influence algebra achievement. The purpose of this study was to test a cognitive model of algebra achievement among undergraduate college students. Algebra achievement was defined as the ability to manipulate algebraic expressions which is a substantial part of many algebra curriculums. The model included cognitive factors that past research has shown relate to overall math achievement. Other goals were to compare a cognitive model of algebra achievement with a model of SAT-M performance and to test for gender differences in the model of algebra achievement.

Structural equation modeling was used to test the direct and indirect effects of algebra experience, working memory, 3D spatial abilities, and computational fluency on algebra achievement. Algebra experience had the strongest direct effect on algebra achievement. Combined direct and indirect effects of computational fluency were as strong as the direct effect of algebra experience. While 3D spatial abilities had a direct effect on algebra achievement, working memory did not. Working memory did have a

direct effect on computational fluency and 3D spatial abilities. The total effects of 3D spatial abilities and working memory on algebra achievement were moderate.

There were differences in the cognitive models of algebra achievement and SAT-M. SAT-M scores were highly related to 3D spatial abilities, but moderately related to algebra experience. There were also gender differences in the cognitive model of algebra achievement. Working memory was highly related to computational fluency for males, but was not related to computational fluency for females.

This study adds to the large body of evidence that working memory plays a role in computational abilities throughout development. The evidence that working memory affects higher level math achievement indirectly through computational fluency and 3D spatial abilities provides clarity to conflicting results in the few studies that have examined the role of working memory in higher level math achievement.

A COGNITIVE MODEL OF ALGEBRA ACHIEVEMENT AMONG UNDERGRADUATE COLLEGE STUDENTS

by
Tammy Daun Tolar

A Dissertation

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in
the College of Education
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ABBREVIATIONS

3D	Three Dimensional
IQ	Intelligence Quotient
SAT-M	Standard Achievement Test - Mathematics
SAT-V	Standard Achievement Test - Verbal
SEM	Structural Equation Model
STM	Short Term Memory
WM	Working Memory

Chapter 1

COGNITIVE COMPONENTS OF MATH ACHIEVEMENT

Introduction

Math achievement has been the subject of an array of research studies in recent years partly because of increased recognition that it plays an important role in good educational and financial outcomes for individuals in our society (U.S. Department of Education, 1997). There is an educational and economic divide favoring the math haves (i.e., those who graduate high school with four years of experience in higher level math, including two years of algebra and often a year of calculus) over the math have-nots (i.e., those who either fail the first year of algebra and never go on, or those who never attempt math beyond pre-algebra). Concerns about student math performance in the U.S. relative to other countries have been covered extensively by the media, and the means for addressing this gap have been the subject of much political debate (Schoenfeld, 2004).

As a consequence of research efforts, there is a substantial amount of evidence suggesting that a variety of factors play a role in math achievement, including socioeconomic status, teacher beliefs and abilities, parental involvement, early acceleration, curriculum, and motivation (Ma, 2005a, 2005b; Ma & Kishor, 1997; Mabbott & Bisanz, 2003; Pajares, 1996; Pajares & Graham, 1999; Pajares & Kranzler, 1995; Sjostrom, 2000). Ultimately, to improve student achievement, these social and motivational processes need to be understood and addressed by schools and society, but these factors are also mediated by and interact with the cognitive capacities and abilities

with which students are equipped. Estimates of math disability rates are similar to those of reading disabilities, which has led to a body of research on cognitive factors that influence math achievement (Geary, 1993, 2003; Geary & Hoard, 2001; Jordan, Hanich, & Kaplan, 2003a, 2003b; Jordan, Levine, & Huttenlocher, 1995; Jordan & Montani, 1997; Rourke, 1993; Rourke & Conway, 1997; Swanson & Jerman, 2006). To be most effective, educational programs need to address these cognitive factors.

Currently there is no comprehensive cognitive model or theory of math achievement. This is partly due to the relatively low number of studies that have examined cognitive processes related to mathematical abilities (as compared to studies on reading achievement, for example). The focus of most of these studies has been on testing the relationship between a few cognitive factors and math achievement (e.g., Bull & Johnston, 1997), or on experimental testing of the involvement of cognitive processes in specific math tasks (e.g., Campbell, 1990). This research has provided a foundation for the development of testable models of math achievement, models which include not only direct effects of cognitive factors but also indirect and interactive effects. Because math achievement is a complex construct, similar to reading achievement, it must be examined in a systematic way to be fully understood. The purpose of this review is to identify cognitive factors that are likely key components of any model of math achievement, describe the evidence implicating them as elements in such a model, and discuss future directions for research necessary to the development of cognitive models of math achievement.

The cognitive abilities with the strongest and most robust correlations to math achievement are general intelligence and fluid reasoning (Bull, Johnston, & Roy, 1999;

Floyd, Evans, & McGrew, 2003), verbal abilities (Cirino, Morris, & Morris, 2002), reading comprehension (Bull & Johnston, 1997; Friedman, 1995), and general knowledge (Batchelor, Gray, & Dean, 1990; Cirino et al., 2002; Floyd et al., 2003). Measures of these abilities serve as useful predictors of math achievement, but these higher level abilities are the result of the coordination of lower level domain specific skills and domain general abilities and resources. It is these processes that have been the focus of much of the research on the cognitive factors involved in math achievement. Therefore, the focus of this review will be on lower level processes and skills that are likely critical to math achievement.

There are four cognitive processes that appear to have special relevance to math achievement: working memory, processing speed, visual spatial abilities, and computational fluency. In evaluating the relationship between each of these processes and math achievement, there are several complexities, both within the constructs and in the contexts in which they operate, that must be considered. Two of these complexities are the way in which math achievement is defined and the age of the individuals being assessed. Another is the degree to which each of these constructs represents domain general and domain specific processes.

The outcome variables in cognitive studies of math achievement range from single digit math fact retrieval to arithmetical computations and reasoning to math problem solving that often incorporates algebraic and geometric reasoning (e.g., Bull & Johnston, 1997; Gathercole, Pickering, Knight, & Stegmann, 2004; Reuhkala, 2001; Rohde & Thompson, 2007). There are qualitative differences between some of these math domains as well as differences in the degree to which some domains of math

incorporate others. This is likely to influence the strength of the relationship between various cognitive factors and math achievement. Any assessment of the literature must account for this possibility.

Most of the cognitive studies of math achievement have focused on elementary-age children, although there are a growing number of studies of math achievement among adolescents and adults. The way in which processes interact to influence math achievement among children is likely to be different than the way in which they interact to influence math achievement among adolescents and adults. One reason for this is that there are qualitative differences in the way in which young children, adolescents, and adults process the same information (e.g., very young children are less likely than older children and adults to verbally encode visual information, see Logie, 1995). There are also developmental differences in cognitive capacities which may influence the degree of the relationship between a cognitive factor and math achievement. Finally, the definition of math achievement is different for children, adolescents and adults. Lower level math skills that are outcomes for children become mediating factors for adolescents and adults. Not only must math domain and developmental differences be considered in evaluating the research on math achievement, but possible interactions between these two factors must also be considered.

Finally, all the cognitive constructs that are the focus of this review include domain general and domain specific processes. The line between what constitutes domain general and domain specific is not well-defined and the research evidence does not provide clear support as to the degree to which these constructs represent domain general and domain specific abilities. There are also likely developmental differences in the

degree to which cognitive factors tax domain general and domain specific processes. A goal of this review is to account for some of the ways in which these processes may interact within and across cognitive constructs in influencing math achievement.

Working Memory and Math Achievement

Working memory capacity is consistently and robustly correlated with math achievement. Correlations between working memory capacity and math achievement have been found in a variety of studies that cover most developmental stages including preschool and elementary age children, adolescents, and younger and older adults (Bull & Johnston, 1997; Bull et al., 1999; Bull & Scerif, 2001; Demetriou, Christou, Spanoudis, & Platsidou, 2002; DeStefano & LeFevre, 2004; Duverne, Lemaire, & Michel, 2003; Espy *et al.*, 2004; Gathercole et al., 2004; Jarvis & Gathercole, 2003; Lehto, 1995; Reuhkala, 2001). Working memory capacity is also correlated with performance across math skills and domains including arithmetic calculations, math reasoning, algebra word problems, and broad assessments of math ability, which include algebraic and geometric reasoning (Dark & Benbow, 1990, 1991; DeStefano & LeFevre, 2004; Engle, Tuholski, Laughlin, & Conway, 1999b; Gathercole et al., 2004; Lee, Ng, Ng, & Lim, 2004). In addition, working memory capacity is related to math performance across populations including students with math disabilities, students with average abilities, and those gifted in math (Bull & Johnston, 1997; Bull et al., 1999; Bull & Scerif, 2001; Dark & Benbow, 1990, 1991; Gathercole et al., 2004; Jarvis & Gathercole, 2003; Swanson & Jerman, 2006; Swanson & Sachse-Lee, 2001). Correlations between working memory capacity and math achievement are robust because working memory capacity accounts for unique variance in math ability, even when controlling for higher level cognitive abilities such as

general intelligence and reading achievement (Bull & Johnston, 1997; Bull et al., 1999; Floyd et al., 2003). The hypothesis that working memory is a key component of math performance is supported by a convergence of evidence from experimental, correlational, neurocognitive, and neuropsychological studies (Burbaud *et al.*, 1995; DeStefano & LeFevre, 2004; Gathercole et al., 2004; Kaufmann, 2002; Kaufmann, Lochy, Drexler, & Semenza, 2004). Based on all this empirical support, it would seem that the connection between working memory and math achievement is a fait accompli; however, working memory is a complex construct which encompasses both domain specific and domain general processes (Baddeley & Logie, 1999; Conway *et al.*, 2005; Oberauer, Heinz-Martin, Wilhelm, & Werner, 2003). It is in examining these complexities that the relationship between working memory and math ability becomes less straightforward.

A Model of Working Memory

Working memory is a cognitive process in which information is maintained in an active state, while that information or other information is being processed. Baddeley's model of working memory has been the implicit if not explicit theoretical framework for most studies of the relationship between working memory and math achievement (e.g., see DeStefano & LeFevre, 2004). According to Baddeley and colleagues (Baddeley, 1996, 2000, 2002; Baddeley & Hitch, 1974; Baddeley & Logie, 1999), working memory is a multi-component structure consisting of two domain specific stores, the phonological loop and the visual-spatial sketchpad, and a domain general control mechanism, the central executive.

The phonological loop is responsible for temporary storage of verbal information, the maintenance of which is aided by a rehearsal mechanism (Baddeley & Logie, 1999).

Capacity for short-term maintenance of verbal information is influenced by the speed with which verbal material can be articulated and strategies for "chunking" information (e.g., "abc" treated as one piece of verbal information instead of three separate letters). These in turn are influenced by development of domain general processes such as processing speed and by domain specific experiences and abilities.

The visual-spatial sketchpad is associated with temporary storage of visual patterns and spatial locations (Baddeley & Logie, 1999; Logie, 1995). There is less empirical evidence concerning the functioning of the visual-spatial sketchpad, and as a consequence, a rehearsal mechanism has not been identified if one exists. The possible *lack* of a rehearsal mechanism may result in a stronger relationship between the visual-spatial sketchpad and the central executive than between the phonological loop and the central executive, although this is conjecture based on limited evidence (Miyake, Friedman, Rettinger, Shah, & Hegarty, 2001).

The central executive is theoretically responsible for a variety of processes including the coordination of information in the domain specific stores and long-term memory (Baddeley, 1996). It is also associated with executive functions such as selective or controlled attention and inhibition. The central executive may also control another component, the episodic buffer, which has recently been added to Baddeley's model and may have some relevance to future research on math achievement because of its purported role in integrating information from the domain specific stores and long-term memory (Baddeley, 2000). However, there is very little theoretical detail or empirical evidence related to this aspect of Baddeley's model. There is enough empirical evidence

concerning the other components, though, to form conclusions and develop hypotheses about the ways in which working memory influence math achievement.

Components of Working Memory and Math Achievement

Whereas working memory capacity is consistently and robustly correlated with math achievement, short-term memory capacities appear to be related to math abilities in only some contexts and often do not account for unique variance in math achievement when controlling for other factors such as reading comprehension and fluid intelligence (Bull & Johnston, 1997; Floyd et al., 2003; Swanson & Jerman, 2006). In fact, some researchers have concluded that it is the central executive or central executive processes that drive the robust relationship between working memory and math achievement (Dark & Benbow, 1990, 1991; Jarvis & Gathercole, 2003; Swanson & Jerman, 2006). For example, a meta-analysis of cognitive predictors of math disabilities revealed that only verbal working memory capacity was a significant predictor of effect sizes (i.e., group differences in math performance between children diagnosed with math disabilities and those identified as average math achievers) when controlling for other factors. These results were based on a regression model that included, among other cognitive variables, verbal short-term memory (STM) span (i.e., phonological loop capacity) and visual-spatial working memory capacity (Swanson & Jerman, 2006). The authors concluded that deficits in "controlled attention to verbal information (i.e., with the influence of other variables such as STM partialed out) was a defining feature of MD children when compared with average achievers" (p. 269).

Based on the reasoning of Swanson and Jerman (2006), central executive processes (e.g., controlled attention) are likely a "defining feature" of math achievement

among average achievers as well. Floyd et al. (2003) conducted a study of math calculation and math reasoning achievement among a large sample of children and adolescents who ranged in age from 6 to 19 years. They tested regression models for each age group using Cattell-Horn-Carroll cognitive ability clusters as predictors. Across age groups, general comprehension and knowledge and fluid reasoning had the highest regression coefficients, followed by working memory. However, short-term verbal memory had relatively low coefficients. Similar results which favor central executive processes over domain specific storage as critical to math performance are found among gifted adolescents (Dark & Benbow, 1990, 1991) and adults (DeStefano & LeFevre, 2004).

This does not mean that domain specific storage capacities do not influence math achievement. Clearly they do (i.e., short-term storage is a fundamental component of working memory so it must necessarily play a role in math achievement), but the degree to which either verbal or visual-spatial short-term memory contributes to math performance seems to be very much dependent on individual experiences and strategies which may allow for more efficient chunking of information in specific domains (Dark & Benbow, 1990, 1991; DeStefano & LeFevre, 2004). For example, whether or not the phonological loop is implicated in low level skills like single-digit calculations among educated adults depends on the relative distribution of counting versus direct retrieval strategies used within the sample being tested (DeStefano & LeFevre, 2004). Counting strategies appear to tax the phonological loop as well as the central executive whereas direct retrieval strategies are more likely to involve the central executive than the phonological loop. In addition, involvement of particular domain specific storage

components also depends on presentation format of the material and the math skills or domain being tested. However, when controlling for these type of factors, it appears the central executive aspects of working memory may be the key predictor of math achievement.

A Controlled Attention Theory of Working Memory Capacity

One way of interpreting results which suggest that domain general central executive processes play a more prominent role than domain specific stores in predicting math achievement is from a theoretical perspective which identifies controlled attention as the chief constraint of working memory capacity (Cowan, 2000, 2001; Engle, 2002). Engle and colleagues (Conway et al., 2005; Engle, 2002; Engle, Kane, & Tuholski, 1999a; Kane, Bleckley, Conway, & Engle, 2001) have argued that one of the key predictors of higher level thinking, in fact, one of the key predictors of performance in many aspects of life, is the capacity for active maintenance of information in the context of potential interference from either external or internal sources of distraction. Although domain specific aspects of working memory (e.g., verbal or visual-spatial short-term storage) influence performance, it is the ability to control attention and inhibit irrelevant stimuli that consistently distinguishes high performers from low ones across cognitive domains. In terms of Baddeley's model, Engle and colleagues identify their definition of working memory capacity with central executive processes, or more specifically, attentional and inhibitory processes associated with the central executive. Empirical evidence to support this theory comes primarily from studies in which working memory capacity is operationally defined in terms of tasks (e.g., operation span, reading span, counting span) which include an information processing component (e.g., making

decisions about the accuracy of single digit calculations or the comprehensibility of sentences) and a memory component (e.g., retention and recall of a series of letters or words). The processing task is designed to prevent individuals from using strategies such as rehearsal and chunking and force them to rely on controlled attention and inhibition to keep the letters or words active and to prevent other stimuli from interfering with the maintenance of that information. Engle and colleagues have found that high performers on these types of tasks are better than low performers at tasks that require them to control attention and inhibit prepotent responses (e.g., antisaccade task, Stroop task). In addition, using structural equation modeling (SEM), they have found that working memory capacity predicts higher level cognitive abilities such as math and verbal achievement and general intelligence even when controlling for short-term memory and processing speed (Conway, Cowen, Bunting, Therriault, & Minkoff, 2002; Engle et al., 1999b; Kane *et al.*, 2004). Furthermore, short-term memory does not correlate with general intelligence when controlling for working memory capacity. Engle and colleagues contend that the common variance across working memory measures which is captured through latent variable analyses represents domain general executive attention, and it is this aspect of working memory that is a key predictor of higher level cognitive abilities.

Working Memory Capacity as Controlled Attention and Math Achievement

There is some evidence that working memory capacity as defined within this framework (WMC) is an important predictor of math achievement. The few studies that have used WMC measures similar to those supported by Engle and colleagues have found that working memory capacity predicts math achievement even when controlling for other factors such as short-term memory and general intelligence (Bull et al., 1999;

Bull & Scerif, 2001; Engle et al., 1999b; Lee et al., 2004). Furthermore, consistent with the hypothesis that controlled attention is a key predictor of higher level thinking, executive attention measures (e.g., Wisconsin Card Sorting Task, Stroop) predict math achievement among school age children as well as college students referred for learning difficulties (Batchelor et al., 1990; Cirino et al., 2002). Impairments in inhibitory processes related to working memory have also been identified as contributing to developmental dyscalculia (Kaufmann, 2002; Kaufmann et al., 2004). However, it is unlikely that attentional and inhibitory processes in isolation (i.e., without a STM component) effectively capture the strong relationship between WMC and math achievement. Instead, these processes in conjunction with active maintenance of information (whatever the domain or domains from which that information comes) may be why WMC is such a robust predictor of math achievement. This appears to be true at least among elementary school students.

In a study of math achievement among 7 and 8 year-olds, Bull and Scerif (2001) found that when controlling for perseveration (i.e., a tendency to remain fixed on a strategy even when it is no longer effective) and inhibition, WMC accounted for unique variance in math achievement, but when controlling for WMC neither perseveration nor inhibition were significant contributors to math performance. Finally, based on their studies of math and/or verbally gifted adolescents, Dark and Benbow (1990, 1991) argued that that aspects of working memory involved with the manipulation of information is especially predictive of math ability (i.e., over verbal ability). This suggests that using measures which incorporate temporary storage and manipulation of

information while controlling for domain specific influences may best capture the strong relationship between working memory and math achievement.

Exceptions to the Rule

The evidence just described suggests that working memory is a critical and perhaps the most influential cognitive component in any model of math achievement, however, the majority of research relating working memory to math achievement is based on elementary-aged children and defines math achievement as performance on arithmetical tasks. There are relatively few studies which examine the relationship between working memory and math achievement among adolescents and adults. In most of these studies, working memory is related to math achievement (Demetriou et al., 2002; DeStefano & LeFevre, 2004; Engle et al., 1999b; Floyd et al., 2003; Gathercole et al., 2004; Lehto, 1995; Reuhkala, 2001; Rohde & Thompson, 2007). Two studies are notable exceptions.

In one study of 15- and 16 year-olds, although visual-spatial working memory correlated with math achievement, verbal short-term memory and verbal working memory did not (Reuhkala, 2001). In addition, 3D mental rotation accounted for more variance in math achievement than working memory. This counterintuitive result (at least, based on the bulk of the research) may have been due to the way in which the tasks were administered. The working memory tasks were administered to groups of students instead of individually, which increases the likelihood of domain specific rehearsal and chunking strategies for some participants (Conway et al., 2005). This method of testing working memory is inconsistent with the individually based method used in virtually all other studies.

In a study of college students, when measures of performance IQ, vocabulary, verbal working memory, processing speed, and spatial ability were included in a regression model of SAT-M performance, working memory was the only predictor that was not significantly related to math achievement (Rohde & Thompson, 2007). Processing speed and spatial ability were based on composite scores from multiple measures, but working memory was based on only one task. Composite or factor scores from multiple measures of working memory are more likely to represent domain general executive attention than are individual measures (Conway et al., 2005). In addition, bivariate correlations were not reported, so whether or not working memory correlated with SAT-M performance without controlling for the other factors could not be determined.

There are several possible reasons for these results, related to differences in the ways the constructs were measured. Another explanation that could account for these results involves an interaction between developmental differences in working memory and math domain differences. It could be that verbal working memory tasks are more likely to tax domain general processes among children than among adolescents and adults. Working memory is not fully developed among children and preadolescents, and there is evidence that suggests that development of working memory involves a complex interaction between changes in articulation rate, speed of retrieval of information from long-term memory, and attentional capacity (Cowan, Saults, & Elliot, 1999, 2002; Cowan *et al.*, 2003). Furthermore, children have less well developed strategies for chunking verbal information than do adults. In addition, among children, arithmetical skills are less automated and likely more taxing on domain general processes, which perhaps causes the

robust link between working memory and math achievement. Among older adolescents and adults, well developed, domain specific abilities and strategies may constrain verbal working memory performance as much as domain general abilities, particularly in individual working memory tasks in which there are less likely to be controls for domain specific processes. Similarly, numerical computational abilities may also be constrained as much by domain specific abilities, many of them likely verbal (e.g., see Dehaene & Cohen, 1997) as by domain general abilities, which would result in a strong relationship between verbal working memory and computational abilities among adults. However, the more abstract algebraic and geometric reasoning required in math assessments such as the SAT-M may be especially demanding of domain general resources, and perhaps other cognitive factors besides working memory are more likely to represent domain general resources among adults. The studies of math achievement among older adolescents and adults described earlier suggest that two of these cognitive factors may be processing speed and visual-spatial abilities.

Processing Speed and Math Achievement

Similar to working memory, processing speed is correlated with math achievement in studies that span developmental stages and math domains (Bull & Johnston, 1997; Demetriou et al., 2002; Floyd et al., 2003; Fuchs *et al.*, 2006; Rohde & Thompson, 2007; Swanson & Jerman, 2006; Swanson & Kim, 2007). Also similar to working memory, processing speed often correlates with math achievement, when controlling for other factors. However, in comparison to working memory, there are fewer studies of math achievement that include processing speed as an independent variable and, although working memory is almost invariably included as an independent

variable in studies of processing speed and math achievement, the relatively small number of studies makes it difficult to form definitive conclusions. These studies are almost equally split into those in which processing speed is a stronger predictor of math achievement than working memory (Floyd et al., 2003; Fuchs et al., 2006; Rohde & Thompson, 2007), those in which the reverse is true (Floyd et al., 2003; Swanson & Jerman, 2006), and those in which both explain similar amounts of variance (Demetriou et al., 2002; Floyd et al., 2003; Swanson & Kim, 2007). There are multiple reasons that could account for this pattern of results.

One factor that may be influencing these results is the way in which the constructs are measured. When multiple measures are used to represent both working memory and processing speed and the measures cross domains (e.g., include processing of verbal, numerical, and visual-spatial information) they account for similar amounts of variance in math achievement, when controlling for the common variance between them and when controlling for short-term memory (Demetriou et al., 2002). In the only study in which one measure was used for working memory and multiple measures were used for processing speed, processing speed was related to math achievement, but working memory was not (Rohde & Thompson, 2007). Perhaps in this case, processing speed was more reliably represented than working memory.

Another factor that could partially account for the conflicting results across studies is the age of the participants and the way in which math achievement is defined. Floyd et al. (2003) examined regression models of math computation and math reasoning for each year of development from 6 to 19 years. These models included fluid and crystallized intelligence, processing speed, short-term and working memory, long-term

memory retrieval, auditory processing, and visual-spatial abilities. Among 5- to 14-year-olds, processing speed was highly related to math calculation. Among 15- to 20-year-olds, processing speed was only moderately related to math calculation. Processing speed was moderately related to math reasoning among 5-14 year-olds, but it was not related to math reasoning among 15-20 year-olds. Across all ages and in both domains of math, working memory was moderately related to math achievement. For younger children and lower level math skills, it appears that processing speed is more important than working memory. For older adolescents and adults and higher levels of math, working memory is more important than processing speed. The results across studies are generally consistent with this hypothesis. A notable contradiction to this is the finding that among college students, processing speed is more predictive of SAT-M performance than working memory (Rohde & Thompson, 2007). Spatial abilities were also more predictive than working memory. As already mentioned, this result may be due to the way in which working memory was measured, but it is also possible that processing speed and spatial abilities are, in fact, more robust predictors of higher level math achievement.

Visual-Spatial Abilities and Math Achievement

In general, correlations between visual-spatial abilities (including both manipulation and short-term storage) and math achievement are low to moderate (for review see, Friedman, 1995) and, according to at least one study of elementary children without diagnosed learning disabilities, visual-spatial short-term storage abilities are not related to math achievement (Bull et al., 1999). Besides being relatively low, correlations between visual-spatial abilities and math achievement are not robust. Almost invariably when other cognitive factors including both higher level cognitive abilities (e.g., general

knowledge, reading comprehension) and more domain general abilities (e.g., working memory, processing speed) are partialed out, visual-spatial abilities do not relate to math achievement (Batchelor et al., 1990; Cirino et al., 2002; Floyd et al., 2003; Swanson & Jerman, 2006). The exceptions to this pattern are found in studies of older adolescents and adults with math outcomes that involve higher level math abilities (Reuhkala, 2001; Rohde & Thompson, 2007).

This evidence does not provide much support for the argument that visual-spatial abilities should be included in a cognitive model of math achievement, particularly among children. Studies of children with learning disabilities, though, suggest that visual-spatial abilities plays a subtle, but profound role in math achievement, and that the importance of visual-spatial abilities to math achievement becomes most apparent when there are deficits in these abilities early in development.

Neuropsychological and Neurocognitive Evidence Relating Visual-Spatial Abilities to Math Achievement

Rourke and colleagues (Harnadek & Rourke, 1994; Rourke, 1993; Rourke & Conway, 1997) have profiled two types of developmental disabilities in arithmetic and math reasoning. Children who are identified as having the less severe type of math disability have deficits in auditory and verbal attention and processing. These individuals respond to non-verbal feedback and experience with mathematical tasks. The type of math disability with the more profound consequences is associated with deficits in visual-spatial processing. Children with nonverbal learning disabilities (NLD) have difficulty with visual-spatial organizational, psychomotor, and tactile-perceptual tasks, but have good rote verbal-memory skills. These children have difficulty with novel and complex

tasks and do not do well with nonverbal problem solving or concept formation. Both types of children perform below normal on tests of arithmetic, but NLD children are more impaired than those with verbal deficits. Girls with Turner Syndrome also have developmental visual-spatial deficits in conjunction with math deficits and intact verbal abilities (Collaer, Geffner, Kaufman, Buckingham, & Hines, 2002; Cornoldi, Marconi, & Vecchi, 2001; Mazzocco, 2001; Temple & Carney, 1995).

Rourke and Conway (1997) associated NLD with distributed cerebral impairments in the right hemisphere. Evidence from neurocognitive research is consistent with the view than an association between visual-spatial and numerical processing is at least partly due to processes associated with the right hemisphere. Parts of the posterior superior parietal area, particularly in the right hemisphere, are active during tasks that require quantity manipulations such as number comparison and approximation. Because this area is also associated with visual-spatial tasks, orienting, and mental rotation, Dehaene and colleagues (Dehaene, 2003; Dehaene & Cohen, 1995; Dehaene, Piazza, Pinel, & Cohen, 2003; Dehaene, Piazza, Pinel, & Cohen, 2005) have argued that the contribution of this area to number processing may be due to “number line” like representations, which would involve orienting and visual-spatial processing.

Reconciling the Evidence

The complex and in some ways discordant pattern of results across research paradigms and populations may be due to a variety of factors that interact to influence the strength of the relationship between visual-spatial skills and math achievement. One factor that appears to influence the nature of this relationship is the math abilities being assessed. In general, the higher level the math, the stronger the relationship between

visual-spatial abilities and math achievement (Friedman, 1995; Geary, Hamson, & Hoard, 2000a; Geary, Saults, Liu, & Hoard, 2000b).

For example, Reuhkala (2001) conducted two studies relating visual-spatial abilities to math achievement among high school students (ages 15 to 16 years). In the first study, she compared performance on a national math exam to measures of visual pattern short-term memory (i.e., matrix span), spatial short-term memory (i.e., Corsi block), and three dimensional (3D) rotation. The math exam included algebraic and geometric reasoning tasks as well as mental arithmetic problems. Correlations between the visual-spatial measures and math achievement ranged from .44 to .57. These correlations are higher than what is typically found across studies of math achievement and visual-spatial abilities (e.g., 0.30 to.45, see Friedman, 1995). The second study, which was based on a different sample of students who took the national math test in a different year, resulted in similar correlations between visual-spatial abilities and math achievement.

In addition, in path models from two different studies of math achievement among college students, the direct effect of 3D mental rotation ability on math achievement was higher when the SAT-M was the criterion variable than when arithmetic reasoning was the measure of math achievement (Casey, Nuttall, Pezaris, & Benbow, 1995; Geary et al., 2000b). The path models in these studies were based on different samples and different control variables (i.e., SAT-V, math self-confidence, and geometry grades versus arithmetical computation and IQ). Finally, as described earlier, in the Rohde and Thompson (2007) study of SAT-M performance among college students, the standardized regression coefficient for spatial ability was similar to that for vocabulary

and higher than the coefficients for nonverbal IQ, processing speed, and working memory. There is no study, though, which definitively tests the relationship between visual-spatial skills and math achievement across math domains.

Another factor that influences the strength of the relationship between visual-spatial abilities and math achievement is the type of visual-spatial ability assessed. Three dimensional (3D) spatial visualization and orientation abilities are more highly related to math achievement than two dimensional (2D) spatial or visual discrimination abilities (for review, see Friedman, 1995). For example, in a study of high school students, Reuhkala (2001) tested verbal short-term memory, verbal working memory, visual-spatial short-term memory, and 3D mental rotation in a regression model predicting math achievement. Only 3D mental rotation was a significant predictor of math achievement, and it accounted for 34% of the variance in math achievement. However, in a second study, Reuhkala (2001) found that static visual-spatial short-term memory (i.e., matrix xpan) correlated more highly with math achievement than either dynamic visual-spatial-short term memory (i.e., Corsi block) or 3D mental rotation. Although when all three factors were included in a single regression model, none of the regression coefficients were significant.

The relationship between visual-spatial abilities and math achievement may also be influenced by the degree to which the math domain being assessed relies on visual-spatial processes per se in comparison to the degree to which both the math domain and the visual-spatial skills being assessed rely on domain general executive processes. One possible explanation for the difference in results across the two Reuhkala (2001) studies may be that the different assessments of math achievement used in the studies

emphasized different math domains (e.g., geometric over algebraic reasoning), and different math domains may place different demands on visual abilities. In addition, relationships between different math domains and visual-spatial abilities may be affected by the degree to which they rely on domain general processes.

Evidence already presented suggests that executive processes, particularly controlled attention and inhibition, may be a key predictor of math in general, but especially higher level math. There is also evidence that suggests that visual-spatial processing may be particularly demanding of executive functions, with 3D spatial visualization tasks being more demanding of executive functions than other visual-spatial tasks (Kane et al., 2004; Miyake et al., 2001). This may in some way explain why NLD children experience more profound consequences to cognitive functioning than do children with primarily verbal deficits. The purported distributed right hemisphere dysfunction associated with NLD may affect controlled attention functions that have been attributed to the right hemisphere (Hedden & Gabrieli, 2006) resulting in the deficits in executive functioning and higher order reasoning more likely to be found in children and adolescents with NLD than those with verbal learning disabilities (Fisher, DeLuca, & Rourke, 1997). Furthermore, there is evidence that the right hemisphere is more functionally integrated and global in its processing than the left hemisphere (Rourke & Conway, 1997). This may result in stronger relationships between functions associated with the right hemisphere (e.g., visual-spatial processing, controlled attention, quantitative reasoning) than those associated with the left hemisphere (e.g., verbal processing, attention to verbal material, numerical procedures and semantic information). It may also be why developmental impairments in the right hemisphere have more

profound impacts on later math achievement then developmental impairments in the left hemisphere. The weight of this evidence suggests that the relationship between math achievement and visual-spatial abilities may be at least partly due to the relationships between both these abilities and executive functioning or more specifically to executive attention. However, this hypothesis is highly speculative and based on limited and disparate pieces of evidence. Most studies that have examined both the effects of working memory or executive attention and visual-spatial abilities on math achievement operationalize visual-spatial processing in terms of 2D abilities and/or operationalize math achievement as arithmetical calculation or reasoning abilities, which for the reasons outlined earlier do not maximize the potential for finding strong interrelationships between these three constructs, particularly among average achievers. More studies of higher-level math achievement among adolescents and adults that test structural models which include multiple measures of working memory, processing speed, and 3D spatial abilities are needed in order to disentangle some of these complexities.

In discussing the relationship between working memory, processing speed, visual-spatial abilities and math achievement, the emphasis has been on the hypothesized critical role of domain general processes, at least partly because research evidence suggests that this aspect of these constructs play a pivotal role in math achievement. However, no model of math achievement would be complete without addressing domain specific abilities. Arguably the most important domain specific ability is computational fluency.

Computational Fluency and Math Achievement

Speed and accuracy in single and multi-digit calculations is related to math achievement among preadolescent children, adolescents, and adults (Geary, Liu, Chen,

Saults, & Hoard, 1999; Geary et al., 2000b; Geary & Widaman, 1992; Royer, Tronsky, & Chan, 1999). Early elementary-aged children with low performance on assessments of math achievement are less likely than high performers to retrieve single-digit math facts from long-term memory and more likely to rely on counting strategies to perform these calculations (Bull & Johnston, 1997). When children with low math abilities retrieve math facts directly from memory, they are more likely to make errors than children with high abilities. Deficits in the ability to retrieve math facts from memory is one of the most consistent findings in studies comparing children with math disabilities to children with normal abilities (Barrouillet, Fayol, & Lathuliere, 1997; Geary, 1993; Geary et al., 2000a; Geary & Hoard, 2005; Geary, Hoard, & Hamson, 1999; Jordan et al., 2003a, 2003b; Jordan et al., 1995; Jordan & Montani, 1997). The inability to produce correct solutions to single-digit problems is associated with long term deficits in mathematics (Geary, 1993; Geary & Hoard, 2001). This evidence provides support for the suggestion that "fast math-fact retrieval at an early age provides the foundation for the later development of a broad array of math competencies" (Royer et al., 1999, p. 196). It is not clear from this evidence, though, why rapid retrieval of math facts among children and efficient single and multi-digit computations among adolescents and adults is related to math achievement. It is also not clear if computational fluency is critical to the type of abstract mathematical reasoning that is a substantial part of math achievement among older adolescents and adults, especially when controlling for domain general processes.

Computational fluency is a manifest requirement in many assessments of math achievement, particularly for younger children and pre-adolescents. The suggestion that children who can quickly and accurately perform computations also do well on math

achievement tests in which fast and accurate computations are required is a somewhat circular argument. There are only two studies relating computational fluency to math achievement among adults, but these studies suggest that computational fluency may be less related to math assessments that include more abstract and symbolic mathematical reasoning than assessments that are more directly reliant on arithmetical calculations and reasoning.

In one study, computational fluency was related to arithmetical reasoning among college students in a structural equation model of math achievement which also included spatial cognition and IQ as predictors (Geary et al., 2000b). The two arithmetical reasoning tests consisted of arithmetic word problems. One test required arithmetical calculations, but the other required only indications of the orders of operations needed for solutions to word problems. Computational fluency was based on two measures, complex addition (e.g., 19 + 8 + 27) and complex subtraction (e.g., 78 – 9). Computational fluency had stronger effects ($\beta = .61$) on arithmetical reasoning than 3D spatial abilities ($\beta = .13$) or IQ ($\beta = .22$). In the second study of adults, computational fluency was compared to performance on the SAT-M among college students (Royer et al., 1999). Computational fluency was based on aggregates of response time and accuracy scores across measures of triple addition (e.g., 4 + 8 + 3 = ?), triple multiplication (e.g., 5 x 2 x 6 = ?), subtraction (including double-digit minus double-digit), and division (including double and triple-digit dividends). Speed of computations accounted for 21% of the variance in SAT-M scores and accuracy accounted for 8% of the variance. Across these two studies, the effect of computational fluency on SAT-M performance was low compared to its effect

on arithmetic reasoning. This difference may have been greater if IQ and 3D spatial abilities had been included as control variables in the study of SAT-M performance.

Computational fluency plays an important role in math achievement among children, adolescents, and adults when math achievement includes arithmetical reasoning problems. It plays some role in higher level math achievement among adults, but the strength of that relationship does not appear to be as strong as it is for lower level math achievement, although this conjecture is based on a small amount of evidence. These hypotheses, though, do not explain why computational fluency is important to math achievement. It is essential to understand the reasons for the relationship between computational fluency and math achievement because there is evidence that suggests that the relationship between these two constructs may have as much to do with domain general processes as to specific numerical abilities.

Explaining the Relationship Between Computational Fluency and Math Achievement

There are several reasons why computational fluency might be important to math achievement. The most obvious reason is that computations are a component of many math tasks and fast, error-free computations naturally contribute to fast, error-free problem solving. Another possible reason is that efficient arithmetic computational abilities may free up working memory resources so the focus of attention can be on higher level math concepts. Hypothetically, though, the inability to quickly retrieve the solution to 7 x 8 should not prevent a student from grasping the concept that 7 x 8 is 7 added 8 times, nor should it prevent a learner from understanding higher level math concepts. In fact, high performing students can focus on and learn more advanced math

topics even when the efficient production of math facts has not completely developed (Newman, Griffin, & Cole, 1989).

Another reason that computational fluency may be related to math achievement is that high performance on computational fluency tasks may be indicative of a well-developed network of numerical relations stored in long term memory. Although there are a variety of strategies that can be used and are used by both children and adults to perform arithmetic computations, the most efficient methods rely on direct retrieval from long-term memory. A strong semantic base of numerical relations is an important part of the more abstract reasoning about numerical relations that is required in higher level mathematics (English & Halford, 1995). Although experience certainly plays an important role in developing this semantic base, there is evidence that working memory resources contribute not only to the development of that network, but also to the retrieval of numerical information from long-term memory. The potential relationship between computational fluency and working memory is a fourth reason for the relationship between computational fluency and math achievement.

Domain General Processes and Computational Fluency

Geary (1993) suggested that developmental deficits in computational fluency may be related to delays or deficits in working memory, especially those aspects of working memory involving allocation of attentional resources. The inability to inhibit irrelevant associations for entering working memory is associated with deficits in math fact retrieval among children (Barrouillet et al., 1997), and as described earlier, deficits in verbal working memory are characteristic of children with math disabilities (Swanson & Jerman, 2006).

Controlled attention is also related to computational fluency among adults. In a review of experimental studies testing the involvement of working memory in mental arithmetic among adults, DeStefano and LeFevre (2004) found that compared to domain specific storage components, the central executive was most consistently implicated in both the retrieval of single-digit math facts and in multi-digit computations. Furthermore, controlled attention, particularly inhibitory processes, have been associated with rapid retrieval of math facts among adults (Arbuthnott & Campbell, 2000; Campbell, 1990; Campbell & Arbuthnott, 1996).

Campbell (1994, 1995) developed a theory of math fact retrieval that is based on the associative nature of semantic information in memory. According to the Network-Interference Model, single-digit math facts are represented as nodes in an integrated network of related problems. A stimulus such as 4 + 3 = ? activates multiple nodes, and the degree of activation for a given node depends on the similarity of the stimulus to the node. Similarity is based on the features of the problems such as the operands and operations as well as the magnitude of the problems. For example, the stimulus 4 + 3 = ? may activate representations for 4 x 3 = 12, 4 + 8 = 12, and 4 + 6 = 10 among others. Whether or not the correct representation is selected depends on the relative activation of the correct node to the incorrect nodes. The role of the ability to inhibit interfering activations is implicit in the model and in the nature of the potentially interfering effects produced by the vast number of associations between an extremely limited pool of items (i.e., ten digits and four operations). In conjunction with supporting the Network-Interference Model, researchers have produced empirical evidence for the role of inhibitory processes in math fact retrieval (Arbuthnott & Campbell, 2000; Campbell,

1990; Campbell & Arbuthnott, 1996). However, in these studies, inferences about the role of inhibitory processes in math fact retrieval are based on error analysis, not on a direct comparison between measures of executive attention or inhibition and computational fluency.

Another domain general process associated with computational fluency is processing speed. As described earlier, processing speed is highly related to calculation abilities among children and moderately related to calculation abilities among adolescents and adults (Floyd et al., 2003). However, the relationship between speed of processing and computational fluency has received little research attention even though the involvement of working memory in computational fluency as well as the emphasis on speed in computational fluency tasks strongly implicates a role for processing speed.

Altogether, the evidence suggests that computational fluency results from a complex interaction of facility with quantitative relationships, semantic knowledge of numerical relations, and more domain general processes. Theoretically, if domain general processes such as controlled attention and processing speed are key contributors to computational fluency, then the relationship between computational fluency and math achievement may have as much to do with domain general processes as with specific numerical abilities.

Conclusions and Future Directions

Working memory, processing speed, visual-spatial abilities, and computational fluency play key roles in math achievement. The relative importance of each of these constructs to math achievement varies in relation to age and math domain.

For young children, working memory and processing speed appear to be the most important factors in math achievement. The impact of these processes on math achievement among children is twofold. Both of these processes influence computational fluency, and for children, computational fluency is a math achievement outcome as well as a factor in achievement in other domains of math. In addition, working memory and processing speed likely influence math achievement independent of their roles in computational fluency.

The evidence suggests that, among children, visual-spatial abilities play less of a role in math achievement than working memory, processing speed, or computational fluency. However, this may be due to what is emphasized on assessments of math achievement among children. Normal visual-spatial processing in children is indicative of normal right hemisphere functioning which has implications for concept formation and quantity estimation abilities. An emphasis on numerical skills rather than conceptual knowledge and quantitative abilities in assessments of math achievement among children may result in an underestimation of the relationship between visual-spatial abilities and math achievement. Future studies of math achievement among children that include assessments that more clearly separate mathematical conceptual knowledge and quantitative abilities from numerical skills may reveal a stronger role for visual spatial abilities than has been found in past studies.

Working memory and processing speed are also important factors in math achievement among adolescents and adults when math achievement is based on arithmetical calculation and reasoning abilities. When math achievement is based on algebraic and geometric reasoning, visual-spatial abilities play a stronger role than either

working memory or processing speed. This suggests that for adolescents and adults, working memory and processing speed may play more indirect roles in math achievement when lower level arithmetical skills (e.g., computational fluency) are treated as factors in math achievement rather than as outcomes. There is much less research, though, on the cognitive factors involved in math achievement among adolescents and adults than among children. Clearly more studies of cognitive factors in math achievement among adolescents and adults are needed to better understand the relationship between math achievement and the cognitive factors described in this review.

Working memory, processing speed, visual-spatial processing, and computational fluency rely to some extent on domain general processes such as controlled attention and inhibition. In order to determine to what degree each of these processes uniquely contribute to math achievement, it is necessary to test cognitive models that include all of these factors as predictors of math achievement. In addition, the pattern of relationships between these factors and math achievement appear to differ depending on whether the definition of math achievement is based on acquired skill, fluid problem solving abilities, or abstract, symbolic mathematical abilities. Systematic comparisons of cognitive models of math achievement across domains of math are needed to determine if these hypothesized differences are real.

A systematic approach to research on the cognitive factors of math achievement is key to the development of cognitive models of math achievement. Cognitive models of math achievement across developmental levels and math domains are essential for educational programs to have the most impact on the many unresolved issues related to math education.

References

Arbuthnott, K., & Campbell, J. I. D. (2000). Cognitive inhibition in selection and sequential retrieval. *Memory & Cognition, 28*, 331-340.

Baddeley, A. (1996). Exploring the central executive. *The Quarterly Journal of Experimental Psychology, 49A*(1), 5-28.

Baddeley, A. (2000). The episodic buffer: A new component of working memory? *Trends in Cognitive Sciences, 4*, 417-423.

Baddeley, A. (2002). Is working memory still working? *European Psychologist, 7*, 85-97.

Baddeley, A., & Hitch, G. J. (1974). Working memory. In G. A. Bower (Ed.), *Recent advances in learning and motivation*. New York: Academic Press.

Baddeley, A., & Logie, R. H. (1999). Working memory: The multi-component model. In A. Miyake & P. Shah (Eds.), *Models of working memory: Mechanisms of active maintenance and executive control* (pp. 28-61). New York: Cambridge University Press.

Barrouillet, P., Fayol, M., & Lathuliere, E. (1997). Selecting between competitors in multiplication tasks: An explanation of the errors produced by adolescents with learning disabilities. *International Journal of Behavioral Development, 21*, 253-275.

Batchelor, E. S., Gray, J. W., & Dean, R. S. (1990). Empirical testing of a cognitive model to account for neuropsychological functioning underlying arithmetic problem solving. *Journal of Learning Disabilities, 23*(1), 38-42.

Bull, R., & Johnston, R. S. (1997). Children's arithmetical difficulties: Contributions from processing speed, item identification, and short-term memory. *Journal of Experimental Child Psychology, 65*, 1-24.

Bull, R., Johnston, R. S., & Roy, J. A. (1999). Exploring the roles of the visual-spatial sketch pad and central executive in children's arithmetical skills: Views from cognition and developmental neuropsychology. *Developmental Neuropsychology, 15*(3), 421-442.

Bull, R., & Scerif, G. (2001). Executive functioning as a predictor of children's mathematics ability: Inhibition, switching, and working memory. *Developmental Neuropsychology, 19*(3), 273-293.

Burbaud, P., Degreze, P., Lafon, P., Franconi, J.-M., Bouligand, B., Bioulac, B., et al. (1995). Lateralization of prefrontal activation during internal mental calculation: A functional magnetic resonance imaging study. *Journal of Neurophysiology, 74*(5), 2194-2200.

Campbell, J. I. D. (1990). Retrieval inhibition and interference in cognitive arithmetic. *Canadian Journal of Psychology, 44*, 445-464.

Campbell, J. I. D. (1994). Architectures for numerical cognition. *Cognition, 53*, 1-44.

Campbell, J. I. D. (1995). Mechanisms of simple addition and multiplication: A modified network-interference theory and simulation. *Mathematical Cognition, 1*, 121-164.

Campbell, J. I. D., & Arbuthnott, K. (1996). Inhibitory processes in sequential retrieval: Evidence from variable-lag repetition priming. *Brain and Cognition, 30*, 59-80.

Casey, M. B., Nuttall, R. L., Pezaris, E., & Benbow, C. P. (1995). The influence of spatial ability on gender differences in mathematics college entrance test scores across diverse samples. *Developmental Psychology, 31*, 697-705.

Cirino, P. T., Morris, M. K., & Morris, R. D. (2002). Neuropsychological concomitants of calculation skills in college students referred for learning difficulties. *Developmental Neuropsychology, 21*(2), 201-218.

Collaer, M. L., Geffner, M. E., Kaufman, F. R., Buckingham, B., & Hines, M. (2002). Cognitive and behavioral characteristics of turner syndrome: Exploring a role for ovarian hormones in female sexual differentiation. *Hormones and Behavior, 41*, 139-155.

Conway, A. R. A., Cowen, N., Bunting, M. F., Therriault, D. J., & Minkoff, S. R. B. (2002). A latent variable analysis of working memory capacity, short-term memory capacity, processing speed, and general fluid intelligence. *Intelligence, 30*, 163-183.

Conway, A. R. A., Kane, M. J., Bunting, M. F., Hambrick, D. Z., Wilhelm, O., & Engle, R. W. (2005). Working memory span tasks: A methodological review and user's guide. *Psychonomic Bulletin & Review, 12*, 769-786.

Cornoldi, C., Marconi, F., & Vecchi, T. (2001). Visuospatial working memory in turner's system. *Brain and Cognition, 46*(Tennet XI: Theoretical and Experimental Neuropsychology), 90-94.

Cowan, N. (2000). Processing limits of selective attention and working memory: Potential implications for interpreting. *Interpreting, 5*, 117-146.

Cowan, N. (2001). The magical number 4 in short-term memory: A reconsideration of mental storage capacity. *Behavioral and Brain Sciences, 24*, 87-185.

Cowan, N., Saults, S. J., & Elliot, E. M. (1999). The microanalysis of memory span and its development in childhood. *International Journal of Psychology, 34*, 353-358.

Cowan, N., Saults, S. J., & Elliot, E. M. (2002). The search for what is fundamental in the development of working memory. In R. V. Kail & H. W. Reese (Eds.), *Advances in child development and behavior* (Vol. 29). New York: Academic Press.

Cowan, N., Towse, J. N., Hamilton, Z., Saults, S. J., Elliot, E. M., Lacey, J. F., et al. (2003). Children's working memory processes: A response-time analysis. *Journal of Experimental Psychology: General, 132*, 113-132.

Dark, V. J., & Benbow, C. P. (1990). Enhanced problem translation and short-term memory: Components of mathematical talent. *Journal of Educational Psychology, 82*, 420-429.

Dark, V. J., & Benbow, C. P. (1991). Differential enhancement of working memory with mathematical versus verbal precocity. *Journal of Educational Psychology, 83*(1), 48 - 60.

Dehaene, S. (2003). The neural basis of the weber-fechner law: A logarithmic mental number line. *Trends in Cognitive Sciences, 7*, 145-147.

Dehaene, S., & Cohen, L. (1995). Towards an anatomical and functional model of number processing. *Mathematical Cognition, 1*(1), 83-120.

Dehaene, S., & Cohen, L. (1997). Cerebral pathways for calculation: Double dissociation between rote verbal and quantitative knowledge of arithmetic. *Cortex, 33*, 219-250.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2003). Three parietal circuits for number processing. *Cognitive Neuropsychology, 20*, 487-506.

Dehaene, S., Piazza, M., Pinel, P., & Cohen, L. (2005). Three parietal circuits for number processing. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 433-453). New York: Psychology Press.

Demetriou, A., Christou, C., Spanoudis, G., & Platsidou, M. (2002). The development of mental processing: Efficiency, working memory, and thinking. *Monographs of the Society for Research in Child Development, 67*, 1-169.

DeStefano, D., & LeFevre, J.-A. (2004). The role of working memory in mental arithmetic. *European Journal of Cognitive Psychology, 16*, 353-386.

Duverne, S., Lemaire, P., & Michel, B. F. (2003). Alzheimer's disease disrupts arithmetic fact retrieval processes but not arithmetic strategy selection. *Brain and Cognition, 52*, 302-318.

Engle, R. W. (2002). Working memory capacity as executive attention. *Current Directions in Psychological Science, 11*(1), 19-23.

Engle, R. W., Kane, M. J., & Tuholski, S. W. (1999a). Individual differences in working memory capacity and what they tell us about controlled attention, general fluid intelligence, and functions of the prefrontal cortex. In A. Miyake & P. Shah (Eds.), *Models of working memory: Mechanisms of active maintenance and executive control* (pp. 102-134). New York: Cambridge University Press.

Engle, R. W., Tuholski, S. W., Laughlin, J. E., & Conway, A. (1999b). Working memory, short-term memory, and general fluid intelligence: A latent-variable approach. *Journal of Experimental Psychology: General, 128*, 309-331.

English, L. D., & Halford, G. S. (1995). *Mathematical education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum Associates.

Espy, K. A., McDiarmid, M. M., Cwik, M. F., Stalets, M. M., Hamby, A., & Senn, T. E. (2004). The contribution of executive functions to emergent mathematical skills in preschool children. *Developmental Neuropsychology, 26,* 465-486.

Fisher, N. J., DeLuca, J. W., & Rourke, B. P. (1997). Wisconsin card sorting test and halstead category test performance of children and adolescents who exhibit the syndrome of nonverbal learning disabilities. *Child Neuropsychology, 3,* 61-70.

Floyd, R. G., Evans, J. J., & McGrew, K. S. (2003). Relations between measures of cattell-horn-carroll cognitive abilities and mathematics achievement across the school-age years. *Psychology in the Schools, 40*, 155-171.

Friedman, L. (1995). The space factor in mathematics: Gender differences. *Review of Educational Research, 65*, 22-50.

Fuchs, L. S., Fuchs, D., Compton, D. L., Powell, S. R., Seethaler, P. M., Capizzi, A. M., et al. (2006). The cognitive correlates of third-grade skill in arithmetic, algorithmic computation, and arithmetic word problems. *Journal of Educational Psychology*, 29-43.

Gathercole, S. E., Pickering, S. J., Knight, C., & Stegmann, Z. (2004). Working memory skills and educational attainment: Evidence from national curriculum assessments at 7 and 14 years of age. *Applied Cognitive Psychology, 18*, 1-16.

Geary, D. C. (1993). Mathematical disabilities: Cognitive, neuropsychological, and genetic components. *Psychological Bulletin, 114*, 345-362.

Geary, D. C. (2003). Learning disabilities in arithmetic: Problem-solving differences and cognitive deficits. In H. L. Swanson, K. R. Harris & S. Graham (Eds.), *Handbook of learning disabilities* (pp. 199-212). New York, NY: The Guilford Press.

Geary, D. C., Hamson, C. O., & Hoard, M. K. (2000a). Numerical and arithmetical cognition: A longitudinal study of process and concept deficits in children with learning disability. *Journal of Experimental Child Psychology, 77*, 236-263.

Geary, D. C., & Hoard, M. K. (2001). Numerical and arithmetical deficits in learning-disabled children: Relation to dyscalculia and dyslexia. *Aphasiology, 15*(7), 635-647.

Geary, D. C., & Hoard, M. K. (2005). Learning disabilities in arithmetic and mathematics. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 253-267). New York: Psychology Press.

Geary, D. C., Hoard, M. K., & Hamson, C. O. (1999). Numerical and arithmetical cognition: Patterns of functions and deficits in children at risk for mathematical disability. *Journal of Experimental Child Psychology, 74*, 213-239.

Geary, D. C., Liu, F., Chen, G.-P., Saults, S. J., & Hoard, M. K. (1999). Contributions of computional fluency to cross-national differences in arithmetical reasoning abilities. *Journal of Educational Psychology*, 716-719.

Geary, D. C., Saults, S. J., Liu, F., & Hoard, M. K. (2000b). Sex differences in spatial cognition, computational fluency, and arithmetical reasoning. *Journal of Experimental Child Psychology, 77*, 337-353.

Geary, D. C., & Widaman, K. (1992). Numerical cognition: On the convergence of componential and psychometric models. *Intelligence, 16*, 47-80.

Harnadek, M. C. S., & Rourke, B. P. (1994). Principal identifying features of the syndrome of nonverbal learning disabilities in children. *Journal of Learning Disabilities, 27*, 144-154.

Hedden, T., & Gabrieli, J. D. E. (2006). The ebb and flow of attention in the human brain. *Nature Neuroscience, 9*, 863-865.

Jarvis, H. L., & Gathercole, S. E. (2003). Verbal and non-verbal working memory and achievement on national curriculum tests at 11 and 14 years of age. *Educational and Child Psychology, 20*, 123-140.

Jordan, N. C., Hanich, L. B., & Kaplan, D. (2003a). Arithemtic fact mastery in young children: A longitudinal investigation. *Journal of Experimental Child Psychology, 85*, 103-119.

Jordan, N. C., Hanich, L. B., & Kaplan, D. (2003b). A longitudinal study of mathematical competencies in children with specific mathematics difficulties versus children with comorbid mathematics and reading difficulties. *Child Development, 74*, 834-850.

Jordan, N. C., Levine, S. C., & Huttenlocher, J. (1995). Calculation abilities in young children with different patterns of cognitive functioning. *Journal of Learning Disabilities, 28*, 53-64.

Jordan, N. C., & Montani, T. O. (1997). Cognitive arithmetic and problem solving: A comparison of children with specific and general mathematics difficulties. *Journal of Learning Disabilities, 30*, 624-634.

Kane, M. J., Bleckley, M. K., Conway, A. R. A., & Engle, R. W. (2001). A controlled-attention view of working-memory capacity. *Journal of Experimental Psychology: General, 130*, 169-183.

Kane, M. J., Hambrick, D. Z., Tuholski, S. W., Wilhelm, O., Payne, T. W., & Engle, R. W. (2004). The generality of working memory capacity: A latent-variable approach to verbal and visuospatial memory span and reasoning. *Journal of Experimental Psychology: General, 133*, 189-217.

Kaufmann, L. (2002). More evidence for the role of the central executive in retrieving arithmetic facts - a case study of severe developmental dyscalculia. *Journal of Clinical and Experimental Neuropsychology, 24*(3), 302-310.

Kaufmann, L., Lochy, A., Drexler, A., & Semenza, C. (2004). Deficient arithmetic fact retrieval -- storage or access problem? *Neuropsychologia, 42*, 482-496.

Lee, K., Ng, S.-F., Ng, E.-L., & Lim, Z.-Y. (2004). Working memory and literacy as predictors of performance on algebraic word problems. *Journal of Experimental Child Psychology, 89*, 140-158.

Lehto, J. (1995). Working memory and school achievement in the ninth form. *Educational Psychology, 15*(3), 271-283.

Logie, R. H. (1995). *Visuo-spatial working memory*. Hillsdale: Lawrence Erlbaum Associates.

Ma, X. (2005a). Early acceleration of students in mathematics: Does it promote growth and stability of growth in achievement across mathematical areas? *Contemporary Educational Psychology, 30*, 439-460.

Ma, X. (2005b). A longitudinal assessment of early acceleration of students in mathematics on growth in mathematics achievement. *Developmental Review, 25*, 104-132.

Ma, X., & Kishor, N. (1997). Attitude toward self, social factors, and achievement in mathematics: A meta-analytic review. *Educational Psychology Review, 9*(2), 89-120.

Mabbott, D. J., & Bisanz, J. (2003). Developmental change and individual differences in children's multiplication. *Child Development, 74*, 1091-1107.

Mazzocco, M. M. M. (2001). Math learning disability and math ld subtypes: Evidence from studies of turner syndrome, fragile x syndrome, and neurofibromatosis type 1. *Journal of Learning Disabilities, 34*, 520-533.

Miyake, A., Friedman, N. P., Rettinger, D. A., Shah, P., & Hegarty, M. (2001). How are visuospatial working memory, executive functioning, and spatial abilities related? A latent-variable analysis. *Journal of Experimental Psychology: General, 130*, 621-640.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school*. Cambridge: Cambridge University Press.

Oberauer, K., Heinz-Martin, S., Wilhelm, O., & Werner, W. W. (2003). The multiple faces of working memory: Storage, processing, supervision, and coordination. *Intelligence, 31*, 167-193.

Pajares, F. (1996). Self-efficacy beliefs and mathematical problem-solving of gifted students. *Contemporary Educational Psychology, 21*, 325-344.

Pajares, F., & Graham, L. (1999). Self-efficacy, motivation constructs, and mathematics performance of entering middle school students. *Contemporary Educational Psychology, 24*, 124-139.

Pajares, F., & Kranzler, J. (1995). *Role of self-efficacy and general mental ability in mathematical problem-solving: A path analysis.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Reuhkala, M. (2001). Mathematical skills in ninth-graders: Relationships with visuo-spatial abilities and working memory. *Educational Psychology, 21*(4), 387-399.

Rohde, T. E., & Thompson, L. A. (2007). Predicting academinc achievement with cognitive ability. *Intelligence, 35*, 83-92.

Rourke, B. P. (1993). Arithmetic disabilities, specific and otherwise: A neuropsychological perspective. *Journal of Learning Disabilities, 26*, 214-226.

Rourke, B. P., & Conway, J. A. (1997). Disabilities of arithmetic and mathematical reasoning: Perspectives from neurology and neuropsychology. *Journal of Learning Disabilities, 30*(1), 34-46.

Royer, J. M., Tronsky, L. N., & Chan, Y. (1999). Math-fact retrieval as the cognitive mechanism underlying gender differences in math test performance. *Contemporary Educational Psychology, 24*, 181-266.

Schoenfeld, A. H. (2004). Math wars. In B. C. Johnson & W. L. Boyd (Eds.), *Politics of education yearbook.*

Sjostrom, M. P. (2000). *Beliefs and practices of teachers regarding the high failure rate in algebra i.* Unpublished Dissertation, Georgia State University, Atlanta.

Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research, 76*, 249-274.

Swanson, H. L., & Kim, K. (2007). Working memory, short-term memory, and naming speed as predictors of children's mathematical performance. *Intelligence, 35*, 151-168.

Swanson, H. L., & Sachse-Lee, C. (2001). Mathematical problem solving and working memory in children with learning disabilities: Both executive and phonological processes are important. *Journal of Experimental Child Psychology, 79*, 294-321.

Temple, C. M., & Carney, R. A. (1995). Patterns of spatial functioning in turner's syndrome. *Cortex, 31*, 109-118.

U.S. Department of Education. (1997). *Mathematics equals opportunity: White paper prepared for u.S. Secretary of education richard w. Riley* (No. ERIC Document Reproduction Service No. ED415119). Washingtion, D.C.: Department of Education.

Chapter 2

A COGNITIVE MODEL OF ALGEBRA ACHIEVEMENT AMONG UNDERGRADUATE COLLEGE STUDENTS

Introduction

Algebra is a gatekeeper. This sentiment has been expressed in one form or another by a variety of sources (Kaput, 1999; Paul, 2005; Schoenfeld, 2004; Steen, 1999; U.S. Department of Education, 1997). Many students lack proficiency in algebra and this limits their access to educational and economic opportunities. Algebra is the "language" of higher level math and science, and as such, it is a prerequisite for courses such as calculus, chemistry, and physics. These courses are the foundation of programs of study that lead to lucrative job opportunities in engineering and science. For these reasons, it is not surprising that algebra is often perceived as an obstacle to many advantages in our society. What is it about algebra that has made it such an obstacle for so many? Intuitively, the answer to this question is that algebra is hard, and certainly, this is a complaint of many students.

The foundation of algebra is abstract, structural representation of numerical relations and mathematical problems. It is this aspect of algebra that makes it a qualitatively different form of mathematical thinking compared to what most students have encountered in their prior math experiences, and this has implications for the cognitive demands that students may experience. This may also be one reason why

algebra is the point at which many students stop in their mathematical development (Ma, 2005a, 2005b; U.S. Department of Education, 1997).

There is a large body of research concerning the type of errors students make in solving algebra problems, the ways in which students conceptualize algebra, and interventions to improve achievement in algebra (Blanton & Kaput, 2002; Carry, Lewis, & Bernard, 1980; Heid, 1996a, 1996b, 2002; Kaput, 1995a, 1995b, 1999; Kieran, 1990, 1992; Kieran, Boileau, & Garancon, 1996; Sfard, 1995; Sfard & Linchevski, 1994; Sleeman, 1984, 1986). There is little research, though, which identifies the underlying cognitive processes that may be related to algebra achievement. Although there is a growing body of literature concerning cognitive processes involved in other domains of mathematics (Geary, 1993; Swanson & Jerman, 2006), most of this research concerns arithmetical computation and problem solving among children. These studies may provide clues as to the cognitive factors likely involved in algebra achievement, but they do not provide definitive evidence of what those factors are or the ways in which they interact to support algebra achievement.

In addition to the qualitative differences between algebra and the mathematics on which most studies are based, there are also developmental differences between the populations in these studies and the adolescents and young adults most impacted by the demands of algebra. The few cognitive studies relevant to higher level math ability among adolescents and adults have used broad math achievement measures as outcomes (e.g., SAT-M, Rohde & Thompson, 2007). The cognitive processes that are related to the types of mathematical problem solving found on achievement tests such as the SAT may differ from processes related to symbolic representation and manipulation which is a

substantial part of many algebra curriculums. Therefore, the goal of this study was to examine a cognitive model of algebra achievement among undergraduate college students to determine if the cognitive processes that have been identified as particularly important in other domains of math achievement are also related to algebra achievement.

Working memory, visual-spatial processing, and computational fluency are three cognitive factors that are important to overall math achievement among adolescents and young adults (Cirino, Morris, & Morris, 2002; Engle, Tuholski, Laughlin, & Conway, 1999; Floyd, Evans, & McGrew, 2003; Gathercole, Pickering, Knight, & Stegmann, 2004; Geary, Saults, Liu, & Hoard, 2000; Reuhkala, 2001; Rohde & Thompson, 2007; Royer, Tronsky, & Chan, 1999). Evidence suggests that these factors may also be important to algebra achievement.

Algebra Achievement and Working Memory

Working memory is a cognitive process in which information is maintained in an active state while that information or other information is being processed (Baddeley & Logie, 1999; Conway *et al.*, 2005; Oberauer, Heinz-Martin, Wilhelm, & Werner, 2003). Short-term memory or the temporary storage of domain specific information (e.g., verbal, visual-spatial) is a part of working memory, but it is the theoretically domain general attentional resources required for the concurrent manipulation of information that appear to cause working memory to be a more robust predictor than short-term memory of performance on complex cognitive tasks such as abstract reasoning and math achievement tests (Engle, 2002; Engle et al., 1999; Floyd et al., 2003; Swanson & Jerman, 2006).

Across studies of children, verbal working memory consistently correlates with performance on arithmetical computation and problem solving tasks (Bull & Johnston, 1997; Bull, Johnston, & Roy, 1999; Bull & Scerif, 2001; Demetriou, Christou, Spanoudis, & Platsidou, 2002; Espy *et al.*, 2004; Gathercole et al., 2004; Jarvis & Gathercole, 2003; Lee, Ng, Ng, & Lim, 2004; Lehto, 1995; Reuhkala, 2001). When controlling for a variety of cognitive abilities including processing speed, short-term memory, and visual-spatial working memory, deficits in verbal working memory appear to be the sole distinguishing characteristic when comparing children with math disabilities to those with average math abilities (Swanson & Jerman, 2006).

Among adolescents and adults, evidence concerning the relationship between verbal working memory and math achievement is inconsistent. When controlling for other factors such as fluid intelligence, long term memory, and visual-spatial abilities, verbal working memory is a predictor of math achievement when the achievement measures are standardized assessments of calculation abilities (e.g., Woodcock Johnson calculations) or arithmetical reasoning abilities (Cirino et al., 2002; Demetriou et al., 2002; Floyd et al., 2003). Differences in verbal working memory capacity are also found among low, medium, and high math ability adolescents based on performance on national math achievement exams (Gathercole et al., 2004). Verbal working memory is a significant predictor of SAT-M performance among college students. However, when controlling for fluid intelligence, vocabulary level, processing speed, and spatial ability, verbal working memory is not related to SAT-M performance (Rohde & Thompson, 2007). Finally, Reuhkala (2001) conducted two studies to compare visual-spatial working memory and other visual-spatial abilities to math achievement among high school

students (ages 15 to 16 years). Math achievement was based on performance on national math exams which included algebraic and geometric reasoning tasks as well as mental arithmetic problems. In both studies, visual-spatial working memory correlated with math achievement. However, the second study also included verbal working memory and short-term memory tasks, neither of which correlated with math achievement.

One explanation for the different outcomes in the studies of adolescents and adults is that working memory may be more of a factor in some math skills than others. In these studies, math achievement was based on assessments that incorporated a variety of math skills including arithmetical, algebraic, and geometric calculation and reasoning abilities. It is impossible to determine which, if any, of these skills may have affected the various outcomes of these studies.

Evidence based on qualitative studies suggests that algebra achievement may be highly related to working memory (Kieran, 1990; Sfard & Linchevski, 1994). Algebraic problem solving requires the ability to transition between arithmetical and algebraic methods and to flexibly switch between operational and structural views of mathematics. Both of these processes may be especially demanding of working memory resources.

Students have a tendency to rely on intuitive, arithmetical methods for problem solving (e.g., guess and check) even when they are taught more effective and efficient algebraic methods (Kieran, 1990, 1992; Nathan & Koedinger, 2000). One possible reason for this is that there are discontinuities between arithmetic and algebra that require students to change preexisting knowledge structures (Kieran, 1990). For example, to many students, the equal sign is a signal to perform an operation or to find the answer. One symptom of this view is the tendency of students to write statements such as 6 + 3 =

9 - 5 = 4. They do not appreciate the representation of equivalence the equal sign engenders. The concept of equivalence is essential to forming a structural view of an equation. Without it students have difficulty accepting expressions such as 3 + 5 = 2 + 6 or solving problems such as 3 + 5 = 2 + ___.

Other discontinuities between arithmetic and algebra include the meaning of letters, intuitive versus formal approaches to problem solving, procedural versus structural orientation, and solving word problems versus representing them (English & Sharry, 1996; Herscovics & Linchevski, 1994; Johanning, 2004; Kieran, 1990; Swafford & Langrall, 2000). There is evidence that the influence of these discontinuities continue into adulthood. For example, undergraduate college students do not perform as well solving equations when prior arithmetic knowledge (e.g., equal sign means 'total') is activated (McNeil & Alibali, 2005).

In addition to the effects of discontinuities between arithmetic and algebra, thinking algebraically may be demanding of working memory because the process of abstracting mathematical structures and operating on them demands the ability to actively maintain multiple conceptions of mathematical objects, to flexibly shift attention between an operational and a structural perspective, and to inhibit a tendency to be influenced by irrelevant surface features of problems (Mason, 1989).

Students who have an operational conception of algebra view algebraic expressions and equations as a series of operations to be performed, and do not recognize structural aspects of the problem that would simplify the solution process (Sfard, 1995, 2000; Sfard & Linchevski, 1994). For example, in solving the equation 2x + 1 = 5, these students would see the solution process as a series of steps (e.g., first subtract 1 from both

sides, then divide by 2). Although two equations might be structurally similar (e.g., $2x + 1 = 6$ and $2(x + 1) = 6$), they would be treated as unique problems. Instead of appreciating that x in the first equation and x+1 in the second equation could be treated similarly in solving these equations, the procedural student would distribute the 2 in the second equation instead of using the simpler approach inherent in the structure of the problem (i.e., divide both sides by 2). As students develop a structural conception of algebra, they begin to appreciate that 2x and $2(x+1)$ and $2(x^2+x+1)$ all represent the product of two numbers.

Students with a fully developed structural conception of algebra can view a series of operations as a mathematical object. The expression 3x+5 does not represent a set of operations to be performed, but is an object itself with a structure and features and one that can be operated on. These students can not only appreciate that problems can be separated into classes based on their structural similarities, but can distinguish between relevant and non relevant features that separate those classes (English & Sharry, 1996). The algebraic thinker must be able to switch between operational and structural views and to apply each view appropriately (Sfard & Linchevski, 1994). Mason (1989) describes this type of mathematical abstraction as a “delicate shift of attention” from seeing an expression as a series of operations to seeing it as an object or property. Working memory may be the mechanism that is most influential in the delicate shift of attention that is crucial to algebraic thinking. One of the goals of this study was to determine if working memory has a direct effect on algebra achievement when controlling for other cognitive factors.

Differences in the measures of math achievement may be one of the causes of conflicting results in studies of the relationship between working memory and math achievement among adolescents and adults, but this was not the only difference across these studies. There were also differences in the ways in which working memory was measured. In the studies in which verbal working memory related to math achievement, multiple measures were used to represent working memory, either mean scores or factor scores or latent variables in structural models (Engle et al., 1999; Floyd et al., 2003; Gathercole et al., 2004). In the two studies in which verbal working memory did not relate to math achievement, single measures were used to represent working memory (Reuhkala, 2001; Rohde & Thompson, 2007). Evidence suggests that the common variance across multiple measures of working memory is more representative of domain general attentional processes than domain specific processes (Conway et al., 2005; Kane et al., 2004).

In one of the studies in which verbal working memory was not related to math achievement (Reuhkala, 2001), the tasks were administered in group settings whereas in most other studies of working memory, working memory tasks are administered on an individual basis. Administering working memory tasks in a group setting may allow more opportunity for domain specific processes (e.g., chunking of verbal material based on associations in long-term memory) to be a factor because individuals who process information more quickly have more time to use these strategies than individuals who process information more slowly (Conway et al., 2005). One goal of this study was to control for the domain specific aspects of working memory capacity by using multiple

measures of working memory and by administering the working memory tasks individually.

Algebra Achievement and 3D Spatial Abilities

Correlations between visual-spatial abilities and math achievement are generally low to moderate (for review, see Friedman, 1995). Among older adolescents and adults, when the visual-spatial tasks include mental manipulation of three dimensional (3D) objects and when the math achievement assessments include higher level math skills than arithmetical computation abilities, the correlations are higher than is typical across studies of math achievement (Casey, Nuttall, & Pezaris, 1997; Friedman, 1995; Reuhkala, 2001; Rohde & Thompson, 2007).

There are two possible reasons for the stronger relationships between visual-spatial abilities and math achievement when the math domain includes higher level math skills than arithmetical computational and reasoning abilities. One reason is that visual-spatial processing is more of a requirement in some domains of higher level math (e.g., geometry) than in arithmetical computation and reasoning. Another reason is that visual-spatial processing, particularly 3D spatial visualization, is highly related to executive functions such as controlled attention (Miyake, Friedman, Rettinger, Shah, & Hegarty, 2001). These domain general processes may be a critical factor in the relatively strong effects of 3D spatial processing on higher level math achievement.

Some aspects of algebra require visual-spatial processing. Algebra achievement includes the ability to graphically represent functional relations. These representations are typically two dimensional, though, and do not require the ability to transform or manipulate 3D representations. Graphical representations are also a relatively small part

of algebra achievement. This suggests that algebraic reasoning may be only moderately related to visual-spatial abilities. However, if domain general processes such as controlled attention are the most important factor in the relationship between 3D spatial abilities and higher level math achievement, then perhaps algebra achievement and 3D spatial abilities are strongly related. A goal of this study was to determine if 3D spatial abilities are related to algebra achievement.

According to Kane et al. (2004), "although spatial storage-rehearsal processes do appear to be more domain general in their predictive power than are corresponding verbal processes, they are not as general or as consistently powerful as the executive-attention processes that are captured by working memory span tasks" (p. 208). Engle and colleagues (Conway et al., 2005; Engle, 2002; Engle et al., 1999) contend that working memory tasks, regardless of the content of those tasks, are highly predictive of higher cognitive processes across domains. There is also evidence that among adults, working memory tasks represent domain specific abilities (e.g., verbal, numeric, visual-spatial) as much as they represent domain general abilities (Perlow, Moore, Kyle, & Killen, 1999; Shah & Miyake, 1996). In order to more fully test the claim that the executive-attention processes captured by working memory tasks, regardless of task content, are more powerful predictors of high level cognitive abilities than the domain general processes associated with visual-spatial abilities, the working memory measures used in this study were based only on verbal content. Any shared variance between working memory tasks with only verbal content and measures of 3D spatial abilities would more likely be due to domain general processes whereas unique variance would more likely be due to domain

specific verbal or visual-spatial abilities. This allowed for more definitive testing of the relationships between these constructs and algebra achievement.

Algebra Achievement and Computational fluency

Computational fluency is defined as fast and accurate arithmetical computations. Speed and accuracy in single and multi-digit calculations are related to math achievement among preadolescent children, adolescents, and adults (Geary, Liu, Chen, Saults, & Hoard, 1999; Geary et al., 2000; Geary & Widaman, 1992; Royer et al., 1999). Evidence suggests that computational fluency may be more related to lower level arithmetical problem solving skills than it is to higher level math achievement, although there are only two studies that have examined the relationship between computational fluency and math achievement among adults. In one study, speed of arithmetic computations accounted for 21% of the variance in SAT-M scores whereas accuracy accounted for only 8% of the variance (Royer et al., 1999). This appears to be a weak relationship compared to the strong effect ($\beta = .61$) computational fluency had on arithmetical reasoning when controlling for 3D spatial abilities and IQ in a study among college students (Geary et al., 2000).

The SAT-M includes arithmetical, geometric, and algebraic reasoning problems. In addition, some problems require procedural-based approaches and others require novel approaches (Gallagher, 1992; Gallagher *et al.*, 2000). It may be that some higher level math skills are more related to computational fluency than others. One of the goals of this study was to determine if algebra achievement is related to computational fluency.

The role of math fluency in algebra achievement is somewhat difficult to predict because of the distinction between fluency with numerical operations and conceptual

understanding of numerical relations. The latter is theoretically critical to algebraic thinking (English & Halford, 1995). The former may not be critical to conceptual understanding of algebra, although, it may be important to fluency in algebraic procedural performance. Another reason that computational fluency may be related to algebra achievement is because similar to algebra achievement; computational fluency appears to be influenced by domain general resources such as controlled attention.

In experimental studies, domain general processes in working memory have been found to be a key component in computational abilities among adults (DeStefano & LeFevre, 2004). More indirect evidence based on error analysis suggests that inhibitory processes play a role in retrieval of single-digit math facts from long-term memory (Campbell, 1990; Campbell & Arbuthnott, 1996). Therefore, another goal of this study was to determine if computational fluency is related to working memory and if computational fluency is related to algebra achievement when controlling for working memory.

Educational Experience and Algebra Achievement

Algebraic conceptual knowledge and skill in algebraic procedures are the products of years of formal education. Evidence suggests that even students in calculus classes do not always exhibit the abstract, structural understanding of numerical relations that is an inherent component of true algebraic thinking (English & Sharry, 1996). However, it is likely that calculus students are more fluent in algebraic manipulations than students who have only had a year or two of algebra. Evidence also suggests that gifted pre-algebra students can exhibit more structural awareness than college students (Dark & Benbow, 1990). Although the effects of cognitive resources and abilities on

algebra achievement are almost certainly mediated by algebra experience, it is possible the combined effects of these cognitive factors may be greater than the effect of experience. Therefore, a goal of this study was to determine if algebra experience mediates the relationship between other cognitive factors and algebra achievement, and if other cognitive factors are related to algebra achievement when controlling for experience.

Study Goals and Hypotheses

The overall purpose of this study was to use structural equation modeling to examine the effects of verbal working memory, 3D spatial visualization, and computational fluency on algebra achievement (see Figure 1). Structural equation modeling is an effective technique for examining cognitive models based on individual differences in performance because both direct and indirect effects of cognitive factors on achievement can be examined. In addition, it allows for the use of multiple measures to represent latent constructs. This is particularly important when trying to capture the domain general processes in working memory. Structural equation modeling is especially useful for testing specific hypotheses about the structural relations between cognitive factors.

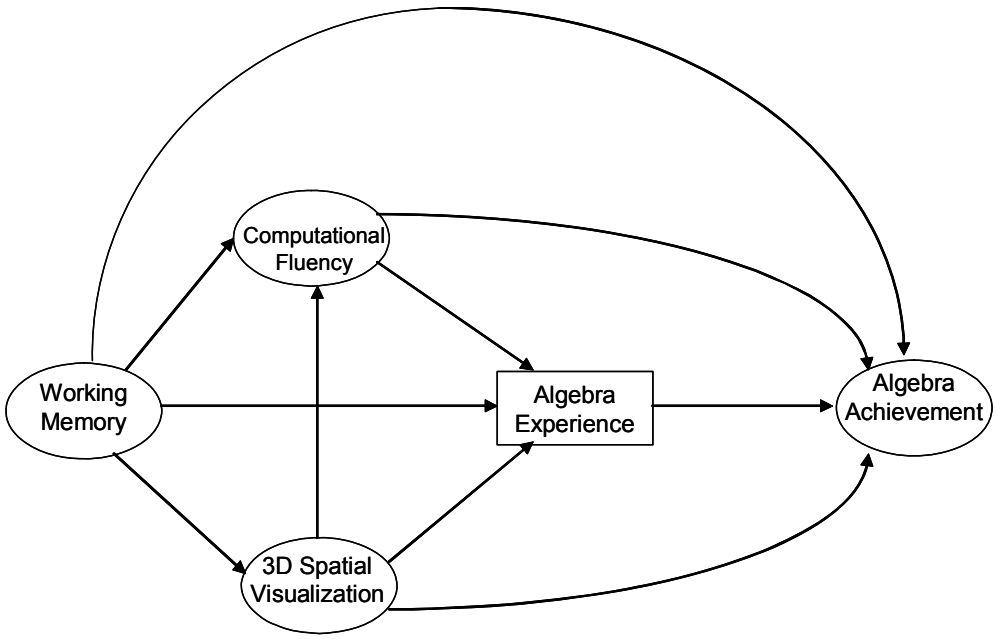


Figure 1. Hypothesized initial model algebra achievement.

For this study, SEM was used to test several hypotheses concerning structural relations in a cognitive model of algebra achievement:

Hypothesis 1. Working memory has a direct effect on algebra achievement when controlling for 3D spatial visualization abilities, computational fluency, and algebra experience.

Hypothesis 2. Working memory has a direct effect on computational fluency, 3D spatial visualization, and algebra experience.

Hypothesis 3. Computational fluency is related to algebra achievement when controlling for the effects of working memory.

Hypothesis 4. 3D Spatial Visualization is not related to either computational fluency or algebra achievement when controlling for working memory.

Based on these hypotheses, Figure 2 shows the final predicted cognitive model of algebra achievement.

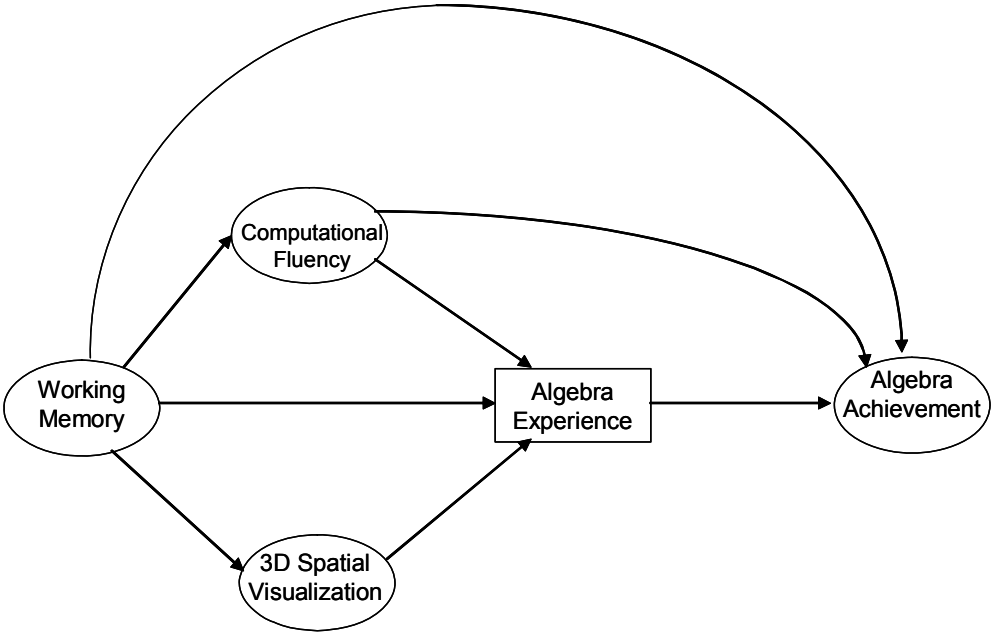


Figure 2. Hypothesized final model of algebra achievement.

Secondary Goals:

Exploratory Models of Math Achievement across Math Domains and Gender

The main purpose of this study was to examine a cognitive model of algebra achievement. Another goal was to determine if the model for algebra achievement held for overall math achievement among adults. There are at least two possible reasons why a cognitive model of algebra achievement might differ from a model for overall math achievement. One is that some domains of math are qualitatively different than others (e.g., geometry is based on spatial relations; algebra is based on numerical relations). A cognitive model for one domain of math might exclude factors relevant to another

domain of math, but a cognitive model of overall math achievement would necessarily have to include factors relevant to all domains of math. Another reason that the cognitive models might be different for algebra achievement and overall achievement is that on tests of overall math achievement (e.g., SAT-M) there is as much emphasis placed on fluid problem solving abilities as there is on textbook procedural knowledge (Gallagher, 1992; Gallagher et al., 2000). Typically, algebra curriculums and assessments of algebra achievement emphasize procedural knowledge. The strengths of the relations between domain specific and domain general abilities and math achievement may differ depending on the degree to which the measure of math achievement emphasizes acquired knowledge versus fluid problem solving abilities.

Working memory, 3D spatial abilities, and computational fluency have each been related to SAT-M performance (Engle et al., 1999; Rohde & Thompson, 2007; Royer et al., 1999). Based on the strength of effects across these studies, 3D spatial relations appears to be more related to SAT-M performance than either working memory or computational fluency. This suggests a different cognitive model than the hypothesized model for algebra achievement. A goal of this study was to determine if the models are different for algebra achievement and SAT-M performance.

Based on studies of math and/or verbally gifted adolescents, Dark and Benbow (1990, 1991) argued that domain general aspects of working memory are especially predictive of math ability over verbal ability. However, there has not been a systematic comparison of cognitive models for math and verbal achievement to determine if there are differences in the strengths of the relations between domain general processes and different domains of achievement. Working memory is related to SAT-V performance

(Engle et al., 1999). There is also evidence that suggests that visual-spatial abilities are more related to performance on the SAT-M than the SAT-V (Rohde & Thompson, 2007). Although it is unlikely that computational fluency is related to SAT-V performance, it is possible, given that working memory has been related to both types of achievement, that computational fluency would have some effect on SAT-V. A goal of this study was to determine if there are differences in the strengths of effects in cognitive models of SAT-V and SAT-M. Similar effects between any of the cognitive factors and both domains of achievement would have implications for the degree to which those constructs represent domain general resources.

A final question addressed by this study was whether or not there are gender differences in the cognitive model of algebra achievement. Although the data is mixed, gender gaps are consistently found in some areas of mathematics and among some populations (Hyde, Fennema, & Lamon, 1990). In comparisons between males and females on measures of complex problem solving, significant differences favoring males emerge in high school and become particularly pronounced among college students and gifted individuals. Males have also outperformed females by an average of 38 points on the mathematics section of the SAT and 70 points on the quantitative section of the GRE (Coley, 2001).

In order to better understand the nature of gender differences in math performance, both computational fluency and visual spatial abilities have been targeted as potential factors that may mediate these differences because gender differences favoring males have also been found in these skills (Casey, 1996; Casey et al., 1997; Casey, Nuttall, Pezaris, & Benbow, 1995; Geary et al., 2000; Royer et al., 1999). In particular,

males consistently outperform females on tasks requiring two and three-dimensional rotations, and these differences tend to be greater among higher ability individuals such as college students (Casey, 1996; Colom, Contreras, Arend, Leal, & Santacreu, 2004; Kerkman, Wise, & Harwood, 2000; Scali, Brownlow, & Hicks, 2000; Voyer, 1996). There is evidence that both computational fluency and 3D spatial visualization mediate gender difference in math performance (Casey et al., 1997; Casey et al., 1995; Geary et al., 2000; Royer et al., 1999). However, examining mean differences in one set of skills to determine if they relate to mean differences in another may explain only part of the gender differences in mathematics performance.

Another way to address this issue is by examining models of math achievement across gender to determine if the pattern of relationships between various cognitive abilities and math achievement differ. This method is relatively unexplored, although, in one study, this method was used to examine gender differences in the relationships between quantitative, verbal, and visual-spatial factors among preschool children and kindergarteners (Robinson, Abbott, & Berninger, 1996). Only the correlation between verbal and spatial factors differed across gender with the correlation being moderate for boys and essentially zero for girls. In an unpublished study (Tolar, 2005), gender differences were found in the relationship between working memory and math achievement. This study was based on 269 college students who had been referred for evaluation because of learning difficulties. A structural model was tested in which working memory and visual spatial abilities were predictors of math achievement. Although there were no gender differences in the significant path coefficient between visual-spatial ability and math achievement, the path coefficient between working

memory and math achievement was significant for the males (.52), but not significant for the females (.21). In addition, the model accounted for 57% of the variance in math achievement for males, but only 43% of the variance for females. This evidence suggests that examining models of math achievement across gender may be a fruitful avenue of research in order to better understand gender differences in math performance. So a final goal of this study was to examine a multi-group structural model of algebra achievement in order to compare females to males in terms of the pattern of relationships between working memory, computational fluency, 3D spatial visualization, and algebra achievement.

In summary, the purpose of the study was to examine a cognitive model of algebra achievement. It was hypothesized that when controlling for algebra experience, working memory would have direct effects on algebra achievement, computational fluency, and 3D spatial abilities. It was also hypothesized that computational fluency, but not 3D spatial abilities would have a direct effect on algebra achievement. It was expected that the total effect of working memory on algebra achievement would be greater than the effect of any of the other cognitive factors. This pattern of results was expected to be unique to algebra achievement in comparison to models of SAT-M and SAT-V performance. Finally, it was predicted that there would be gender differences in the patterns of effects in the cognitive model of algebra achievement.

Methods

Participants

Data was collected from 233 undergraduate college students (112 females and 121 males) recruited from a psychology research participation pool at a large, ethnically diverse, urban university. Students received credit in introductory psychology courses for participation. Thirty-five of the students (15 females and 20 males) did not attend the second of the two testing sessions. The data from three males were excluded, two because they did not meet the age requirements of the study and the third because of errors in data collection. As a result, the participants in this study consisted of 195 students (97 females and 98 males) between the ages of 18 and 25 years (mean = 19.5 , SD = 1.5). All of the students had taken at least one of the following high school or college courses within 5.5 years of study participation: algebra, advanced algebra/trigonometry, pre-calculus, or calculus. None of the students had been diagnosed with a learning disability (based on self-report), although one male reported he had ADHD. See Table 1 for more demographic information.

Table 1

Participant Demographics

Demographic	Females (N = 97)	Males (N = 98)	Total (N = 195)
Mean age (SD)[a]	19.2 (1.5)	19.9 (1.5)	19.5 (1.5)
Age range	18 - 25	18 - 24	18 – 25
College major		Percent	
Physical sciences/mathematics	15	13	14
Social sciences (e.g., psychology)	15	9	12
Applied sciences (e.g., nursing)	10	5	8
Business	14	24	19
Liberal arts	35	34	34
Undecided/no info	9	14	11
Mean number of years since last algebra based class (SD)	1.3 (1.2)	1.5 (1.2)	1.4 (1.2)
Algebra-based course level[a]		Percent	
One algebra course	0	1	1
Two algebra courses	2	2	2
Three or more algebra courses	18	5	11
One or more pre-calculus or advanced algebra/trigonometry course	55	48	51
Calculus I	25	34	29
Calculus II or higher	1	10	6

[a] Females differed from males ($p < .05$).

Apparatus, Materials, and Procedures

The students participated in two 1.5 hour sessions. They were tested in groups of 1 to 12 during the first session and individually during the second session. The time between the two sessions ranged from 1 to 27 days (median = 6 days) with the exception of one student who completed the second session 50 days after the first session.

Two spatial visualization tasks and three algebra achievement tests were given during the first session. All tasks administered during this session were paper-and-pencil with task order counterbalanced across two protocols. Tasks were given in the following order for the first protocol: 3D Mental Rotation Test (MRT) Part 1, AAIMs Algebra Content Test, 3D MRT Part 2, DTMS Elementary Algebra Test, DAT Spatial Relations Test, and Algebra Equations Test. Task order was reversed for the second protocol with the exception of the MRT tests for which Part 1 was given before Part 2 in both protocols. Students were randomly assigned to session protocols. All group sessions were held in classrooms or conference rooms with students seated at desks or around conference tables.

Four working memory tasks, three computational fluency tests, and one algebra achievement test were given during the second session. Students also completed a math background sheet at the end of this session. Two of the working memory tasks were computer-based and the other two were given orally. The math and algebra tasks were pencil-and-paper tests. Similar to the first session, tasks in this session were counterbalanced across two protocols. Tasks were given in the following order for the first protocol: Counting Span, Division Test, Letter-Number Sequence, DTMS Intermediate Algebra Test, Addition/Subtraction Test, Reading Span,

Subtraction/Multiplication Test, and Backwards Digit Span. Task order was reversed for the second protocol. Students were randomly assigned to session protocols with no connection between protocol assignments across the two testing sessions. All individual sessions were held in small rooms or offices with students seated at a large desk or small table. Computers with color monitors were used for the computer-based working memory tasks. Students sat directly facing the computer screens.

Tasks

The measures used in this study were chosen to optimize the likelihood of producing an acceptable SEM measurement model which is a function of the number and quality of the observed measures as well as the sample size. Although three or more measures are recommended to avoid nonconvergence or improper solutions, two measures may be acceptable if they both are likely to load highly on the latent variable (e.g., standardized loadings > .60) and if the sample size is relatively large (e.g., 100 - 150 participants, Kline, 2005). Given these constraints, three measures were chosen to represent computational fluency. Four measures were initially chosen to represent algebra achievement because there is no widely accepted standardized measure of algebra achievement and no established pattern of correlations between measures. However, for theoretical reasons and based on performance by the participants in this study on the algebra measures, subscales from two of the measures were combined as described later, resulting in three observed variables representing the algebra achievement latent variable. Four measures were chosen to represent working memory because there is no universal agreement on the best measures for working memory capacity, particularly when trying to capture the domain general, central executive aspects of working memory. In addition,

correlations among working memory measures tend to be moderate (Conway, Cowen, Bunting, Therriault, & Minkoff, 2002; Engle et al., 1999; Kane *et al.*, 2004). Two measures were chosen to represent 3D spatial visualization because research suggests that 3D spatial visualization measures tend to correlate highly with each other and load highly onto a distinct latent construct (Kane et al., 2004; Miyake et al., 2001).

Algebraic Achievement

DTMS Algebra Skills Tests. Participants took two Descriptive Tests of Mathematics Skills (DTMS, Forms M-K-3LDT), one in Elementary Algebra Skills (The College Board, 1995a) and the second in Intermediate Algebra Skills (The College Board, 1995b).

The Elementary Algebra Skills test consisted of 35 multiple-choice problems that were grouped into one of four clusters: operations on real numbers (e.g., -3 – 2 =), operations with algebraic expressions (e.g., 3(m+4) =), solutions to equations and inequalities (e.g., If t – 2 = 6, then t + 2 =), and applications (e.g., If the sum of 4 numbers is 70, what is the average (arithmetic mean) of the numbers). Reliabilities for this test were relatively high (alternate-forms estimate = .85) based on a diverse sample of two-year (N = 150) and four-year (N = 374) college students.

The Intermediate Algebra Skills test consisted of 30 multiple-choice problems that were also grouped into one of four clusters: algebraic operations (e.g., $36x^2 – 1 =$), solutions of equations and inequalities (e.g., 2x – 7 = 8, 5x + y = 6, in the solution of the system of equations, what is the value of x?), geometry (e.g., given a picture of a triangle with two of the angle measures given and the third represented by x, what is the value of x?), and applications (e.g., probability, interpreting graphs). Reliabilities for this test were

also relatively high (alternate-forms = .73) based on 155 two-year and 251 four-year college students.

Students were allowed up to 30 minutes to complete each test; however, most students stopped working before the end of the time allowed, with many completing the tests in 15 to 20 minutes. When all students in a testing session stopped working, the test was ended. This did not appear to affect performance, because participants in this study performed well above the college students who participated in the test development/reliability studies, mean = 27.1, SD = 5.3 compared to mean = 21.8, SD = 6.5 for the Elementary Algebra Skills Test and mean = 16.9, SD = 5.9 compared to 13.7, SD = 5.3 for the Intermediate Algebra Skills Test (The College Board, 1989). In addition there was a substantial negative skew in performance on the Elementary Algebra Skills Test among the participants in this study.

For this study, the operations with algebraic expressions (9 items) and solution of equations and inequalities (8 items) clusters from the Elementary Algebra Skills test were combined with the algebraic operations (7 items) and solution of equations and inequalities (8 items) clusters from the Intermediate Algebra Skills test to form one measure (32 items). These clusters best represented the algebraic skills that are the focus of this study, that is, symbolic representation and manipulation. The other clusters included many problems that could be solved arithmetically, and in the case of the geometry cluster, specifically represent mathematical skills that are not the focus of this study. Scores on this measure were based on the number of correct items and could range from 0 to 32.

AAIMS Algebra Content Test. This multiple-choice assessment of algebraic skills and abilities was developed in conjunction with Project AAIMS: Algebra Assessment and Instruction - Meeting Standards which is funded by the U.S. Department of Education for the purposes of evaluating "alignment between algebra curriculum, instruction, and assessment for students with and without disabilities" (p. 3, Foegen, Olson, & Perkmen, 2005). Appendix A contains all problems from the AAIMS Algebra Content Test. Reliability and validity results for this measure were based on a sample of 62 high school students (38 grade 9, 20 grade 10, 3 grade 11). Alternate form reliability for this test was .74, and test-retest reliability was .80. This test was composed of problems that represent most of the concepts covered in a typical first-year high school algebra course (Foegen et al., 2005). Participants were given 15 minutes to complete this test. Scores were based on the number of correct items and could range from 0 to 16.

Algebra Equations Test. This free response test was a modified version of an assessment of algebra equation solving skills used in a study of university psychology and engineering students (Carry et al., 1980). Appendix B lists all of the problems from the Algebra Equations test. Eight of the problems come from the Carry, et al. (1980) study. The other four problems are linear equations similar to those found in most standard first year algebra textbooks (e.g., Foster, Winters, Gell, Rath, & Gordon, 1995). Students were allowed 12 minutes for this test. Scores were based on the number of correct items and could range from 0 to 12.

Working Memory Capacity (WMC)

Reading Span (RSPAN, Conway et al., 2005; Unsworth, Heitz, Schrock, & Engle, 2005). This computer-based task was composed of 12 trials with each trial consisting of

two to five letters to be recalled. During a trial, a sentence followed by a question mark and a capital letter appeared in the middle of the computer screen (e.g., The tugboat had never been so in love. ? H). As soon as the sentence appeared on screen, the participant read it out loud, evaluated the meaning for as long as necessary, said “Yes” or “No” depending on whether or not the sentence made sense, then immediately read the letter out loud, and kept the letter in memory. The experimenter advanced the screen to the next item as soon as the participant read the letter out loud. After two to five sentences, three question marks appeared in the middle of the screen which prompted the participant to write down the letters, in order, on a pre-printed form. Instructions for the task were shown on screen and read aloud by the task administrator. Three practice trials (two letters in each trial) were given as part of the instructions. The experimenter recorded the participants’ “YES” or “NO” responses to ensure that they were not sacrificing accuracy on the processing task in order to improve performance on the memory task. Scores for RSPAN were the total number of letters recalled in the correct positions and could range from 0 to 42.

Counting Span (CSPAN, Conway et al., 2005). This task was also computer-based and was composed of 15 trials with each trial consisting of two to six single-digit numbers (3 through 9) to be recalled. A trial began with several randomly placed dark blue circles, dark blue squares, and light green circles appearing on screen. As soon as the objects appeared on screen, the participant counted the dark blue circles out loud (taking as much times as necessary for accuracy), immediately repeated the final total out loud, and memorized the number (see Figure 3 for example).

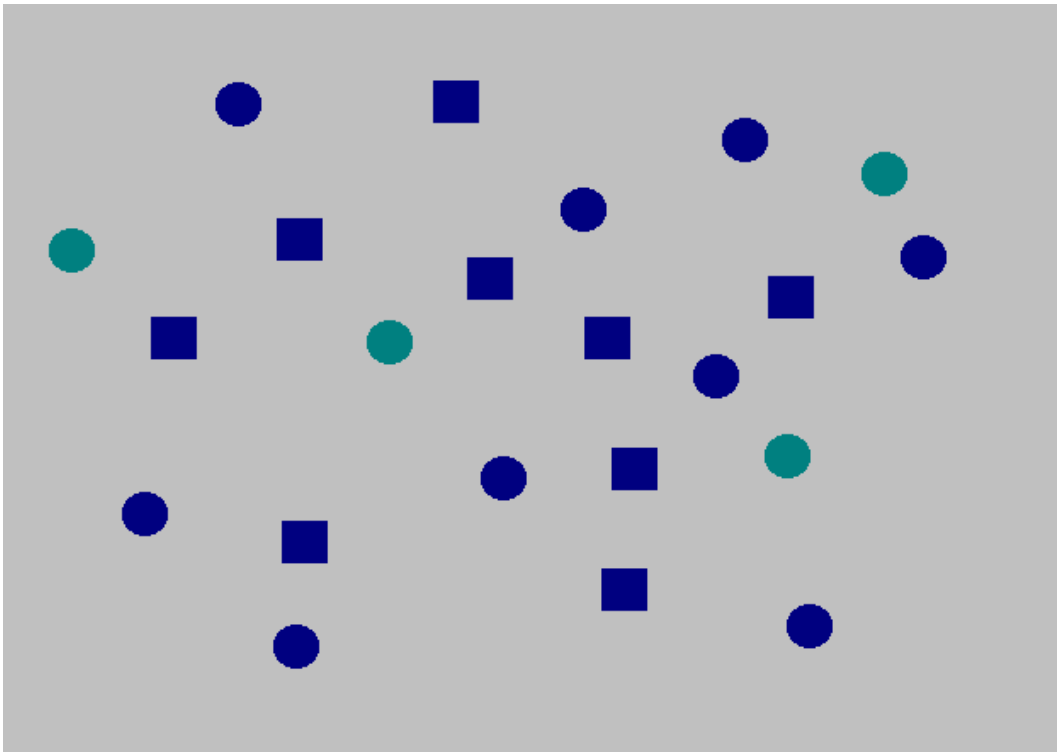


Figure 3. Example item from counting span task.

The experimenter advanced the screen to the next item as soon as the participant repeated the final number out loud. When three question marks appeared in the middle of the screen the participant wrote down the digits, in the order they were originally presented, on a pre-printed form. Instructions for the task were shown on screen and read aloud by the task administrator. Three practice trials (two items in each trial) were given as part of the instructions. Scores for this task were the total number of digits recalled in the correct positions and could range from 0 to 60.

Digit Span - Backwards (DSPANB). The Weschler Adult Intelligence Scale – Third Edition (WAIS-III, Wechsler, 1997) Digits Backward subtest was administered according to the test manual. For each trial, the experimenter read a series of digits (1 through 9) at a rate of about one per second, then the participant repeated the digits out loud in reverse order. A trial consisted of two to eight digits. Two trials for each digit

length were administered until the participant failed to recall the digits in the correct order for both trials of a given length. Scores were based on the number of trials in which the participant correctly recalled the digits and could range from 0 to 14.

Letter-Number Sequence (LNSEQ). The WAIS-III (Wechsler, 1997) Letter-Number Sequencing subtest was administered according to the test manual. For each trial, the experimenter read a series of digits and letters at a rate of about one per second. The participant repeated the items, digits first from lowest to highest, then letters in alphabetical order. A trial consisted of two to eight digits and letters with three trials of each length. The task ended when the participant failed to recall the items in the correct order for all three trials of a given length. Scores were based on the number of trials in which the participant correctly recalled the items and could range from 0 to 21.

3D Spatial Visualization

3D Mental Rotation Test (MRT). The Revised Vandenberg & Kuse Mental Rotations Test (Peters, 1995) is a redrawn version of the Vandenberg and Kuse (1978) MRT and Shepard and Metzler (1978) figures. There were two parts to this test with each part consisting of twelve items. Each item (see Figure 4) consisted of a stimulus figure and four target figures, two of which were rotated versions of the stimulus figure. To correctly complete an item, participants had to identify both of the rotated versions of the stimulus figure. Participants were given three minutes to complete each part of the test. Scores were based on the number of correct items and could range from 0 to 24 points.

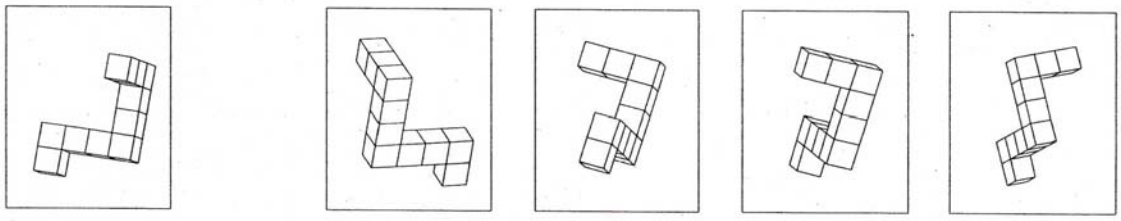


Figure 4. Example item from 3D mental rotations task.

3D Spatial Relations. The DAT Spatial Relations Test (Bennet, Seashore, & Wesman, 1989) consisted of 35 items similar to the one shown in Figure 5. To correctly complete an item, participants had to identify the figure that represented the stimulus figure after it has been folded into its 3D construction. The test directions allowed for 15 minutes to complete as many items as possible, but because of time constraints, participants in this study were given a 10 minute time limit. Scores were based on the number of correct items and could range from 0 to 35.

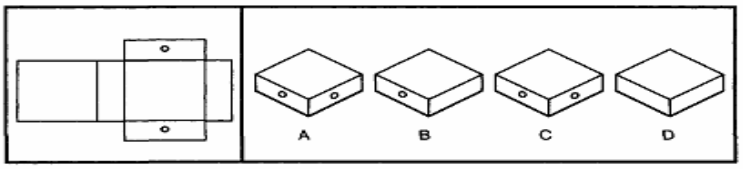


Figure5. Example item from 3D spatial relations task.

Computational fluency

The three tests of computational fluency were part of the Number Facility factor in the ETS Kit of Factor-Referenced Cognitive Tests (Ekstrom, French, Harman, & Derman, 1976). Each test had two parts with two minute time limits for each part. Each part contained 60 items. Only part one of each test was administered in this study. Scores for each of these tests were based on the number of correct items and could range from 0 to 60.

Division Test. All dividends in this test were two or three digits and all divisors were single digits (e.g., 64 ÷ 4, 150 ÷ 6).

Subtraction and Multiplication Test. This tests consisted of alternating rows of subtraction and multiplication problems. Both the minuends and subtrahends of all subtraction problems were two digits (e.g., 98 - 75). The multiplication problems were composed of two digit multiplicands and single digit multipliers (e.g., 86 x 6).

Addition and Subtraction Correction Test. This test consisted of addition and subtraction problems with suggested answers (e.g., 11 + 23 = 34, 35 - 10 = 20). The participants were required to circle the C (correct) or I (incorrect) next to each problem they attempted.

Algebra Experience

Students completed a math background form (see Appendix C) which provided information about the math courses they took in high school and college and the semester and year during which they took each course. Time since last algebra-based course was calculated for each student. Each student was also given an experience score according to the highest level of algebra-based math taken:

1 = one algebra course

2 = two algebra courses

3 = three or more algebra courses

4 = an advanced algebra/trigonometry or pre-calculus course

5 = a calculus I course

6 = a calculus II course

This variable represented a direct measure of math experience at the time algebra achievement was measured, whereas in analyses of SAT performance, this variable was an estimate of the relative math experience of participants at the time they completed the SAT. The highest level of math experience for most of the participants occurred in high school during their senior years which was likely during or just after they had taken the SAT exams. Few students took a higher level of math during college with the exception of all of the students who had taken calculus II or higher and some of the students who had taken calculus.

SAT Math and Verbal Scores

SAT scores were gathered from university records for students who granted permission to access these records (N = 135, 65 female, 70 male). Exploratory analysis was conducted using this smaller sample to compare SEM models for algebra achievement to those for overall math achievement.

Results

Data Screening

A critical assumption of structural equation modeling is multivariate normality; therefore, the data was carefully screened for univariate and multivariate outliers as well as violations of univariate and multivariate normality. For all measures except algebra experience, values that were more than 3 standard deviations from the mean were replaced with values 3 standard deviations from the mean. This affected 5 out of 2340 values. Univariate skew and kurtosis for all variables were at acceptable levels (see Table 2 for descriptive statistics). Examination of leverage, studentized t, and Cook's D values indicated there were no observations unduly influencing multivariate correlations.

Finally, for the 12 continuous variables, Mardia's (1970, 1983) relative multivariate kurtosis value (.99, $p > .10$) and multivariate skew and kurtosis statistic ($\chi^2 = 2.86$, $p = .239$) indicated that it was reasonable to assume multivariate normality.

Table 2

Descriptive Data and Reliability Estimates (Cronbach's alpha) of Observed Variables

Task	M	SD	Range	Skew	Kurtosis	α
Algebra Experience	4.2	0.8	1 – 6	-0.31	1.07	-
AAIMS Algebra Test	10.0	3.2	3 – 16	-0.19	-0.66	0.76
Algebra Equations Test	6.4	2.4	0 – 12	0.21	-0.17	0.71
DTMS Algebra Subtests	19.4	6.2	6 – 32	-0.02	-0.96	0.86
DAT Spatial Relations	18.5	6.2	5 – 34	0.20	-0.51	
3D Mental Rotation	9.1	4.7	0 – 23	0.45	-0.28	
Subtraction/Multiplication Calculations	19.2	6.5	7 - 38.8*	0.39	-0.22	
Division Calculations	11.7	5.4	2 - 28.4*	0.77	0.43	
Addition/Subtraction Correction	31.0	8.7	15 - 57.4*	1.11	1.07	
Digit Span - Backwards	7.7	2.2	3 – 14	0.39	-0.33	
Letter-Number Sequence	12.1	2.6	7 – 20	0.61	0.23	
Counting Span	34.4	10.0	13 - 59	0.15	-0.53	
Reading Span	24.5	6.5	8 – 41	0.06	-0.48	

* Corrected outlier values

Structural Model Analyses

In the following discussion, *measures*, *observed variables*, and *indicators* are used interchangeably and refer to the tests given to the participants (e.g. AAIMs Algebra Test, Reading Span). *Factors*, *latent variables*, and *constructs* are used interchangeably and refer to the abilities hypothetically measured by the tests (e.g., algebra achievement, working memory capacity). *Measurement model* refers to an SEM model in which the observed variables are restricted to load only onto the latent variables they theoretically represent. The latent variables are allowed to correlate freely with each other. If this relatively unrestricted model (i.e., no restrictions imposed on the relationships between

constructs) has a good fit, then a *structural model* with theoretically determined relationships between constructs can be tested.

Three sets of models were analyzed in order to answer three broad questions related to algebra achievement. The first set of models addressed the main hypotheses of this study concerning the effects of working memory, 3D spatial visualization, and computational fluency on algebra achievement. The second set of analyses were conducted to examine potential differences in these effects on SAT-M and SAT-V performance, and the third set of analyses were conducted to examine potential gender differences in the pattern of these effects on algebra achievement. The last two sets of analyses were exploratory because the samples sizes were relatively small and the study design did not support definitive testing of hypotheses related to these questions.

All structural models were analyzed in LISREL (Joreskog & Sorbom, 2006), and model fits were evaluated with indices recommended by Kline (2005). These indices include the model chi-square, the standardized root mean-square residual (SRMR), the Steiger-Lind root mean square error of approximation (RMSEA, Steiger, 1990) with its 90% confidence interval, and the Bentler comparative fit index (CFI, Bentler, 1990). In addition, the chi-square difference statistic was used to evaluate the effects of model trimming.

For model chi-square, a significant result ($p < .05$) indicates a poor model fit. However, this statistic is sensitive to sample size with larger samples increasing the likelihood of a significant result (Kline, 2005). Although problematic, it is traditionally reported in SEM analyses, and it is particularly useful for the purposes of this study in calculating chi-square differences when evaluating the effects of model trimming. During

model trimming, when two models did not differ in goodness of fit (i.e., chi-square difference was not significant, $p > .05$), the more parsimonious model (i.e., fewer paths) was chosen.

RMSEA values closer to 0 indicate better fit (Kline, 2005). For this study, models with RMSEA $\leq .05$ were considered good fits, $\leq .08$ reasonable fits, and $\geq .10$ poor fits. This statistic was evaluated in conjunction with its 90% confidence interval (CI). A CI with a lower bound greater than .05 was considered a poor fit, and although a CI with a lower bound less than .05 was not rejected, if the upper bound was greater than or equal to .10, the possibility of a poor fit was also not rejected.

CFI's $\geq .90$ are acceptable (Kline, 2005), although CFI's $\geq .95$ are more desirable. For this study, the more stringent criterion of .95 was used to assess model fit.

SRMR values closer to 0 indicate better fitting models. For this study, a model with a SRMR $\leq .05$ was considered an excellent fit, although one with a SRMR $\leq .10$ was accepted if the other fit indices indicated an acceptable model fit.

Structural Analyses I: Cognitive Predictors of Algebra Achievement

The pattern of correlations between the 12 observed variables (see Table 3) suggests that these measures were good representations of the four factors of interest. The correlations between indicators within a construct tended to be higher than between indicators across constructs.

Table 3

Correlations between Observed Variables

Variable	1	2	3	4	5	6	7	8	9	10	11	12
1. AAIMS	-											
2. AlgebraEq	***0.64***	-										
3. DTMS	***0.75***	***0.73***	-									
4. DAT	***0.33***	***0.18***	***0.26***	-								
5. MRT	***0.28***	***0.24***	***0.30***	***0.60***	-							
6. Sub/Mult	***0.33***	***0.33***	***0.35***	0.05	0.02	-						
7. Division	***0.40***	***0.40***	***0.42***	0.12	***0.21***	***0.67***	-					
8. Add/Sub	***0.28***	***0.31***	***0.32***	***0.15***	***0.33***	***0.65***	***0.63***	-				
9. DSpanB	***0.14***	0.12	***0.18***	0.10	***0.14***	***0.17***	***0.29***	***0.18***	-			
10. LNSeq	0.08	***0.14***	***0.16***	***0.15***	***0.26***	0.13	***0.30***	***0.31***	***0.51***	-		
11. CSpan	0.09	0.06	0.09	0.04	**0.16**	**0.23**	**0.31**	**0.28**	**0.40**	**0.42**	-	
12. RSpan	0.08	0.12	0.07	0.05	0.06	**0.19**	**0.24**	**0.15**	**0.29**	**0.35**	**0.53**	-

Note. AAIMS = AAIMs algebra test; AlgebraEq = Algebra equations test; DTMS = DTMS elementary and intermediate algebra subtests; DAT = spatial relations test; MRT = 3D mental rotation test; Sub/Mult = subtraction/multiplication computational fluency test; Division = division computational fluency test; Add/Sub = addition/subtraction correction test; DSpanB = Digit span backwards; LNSeq = Letter-number sequence; CSpan = counting span; RSpan = reading span. Shaded areas include correlations between observed variables within a construct. Bolded correlations are significant ($p < .05$).

Measurement models. Figure 6 shows the measurement model that was analyzed in order to more definitively determine if the measures loaded onto the appropriate latent variables. Because algebra experience was represented by a single ordinal variable, the path between the observed and latent variable was set to 1 and the measurement error set to 0. All measures were set to load only onto the latent variables they were selected to represent, all covariances between measurement error terms were set to 0, and all latent

variables were allowed to covary freely with each other (see Table 4 for correlations between factors).

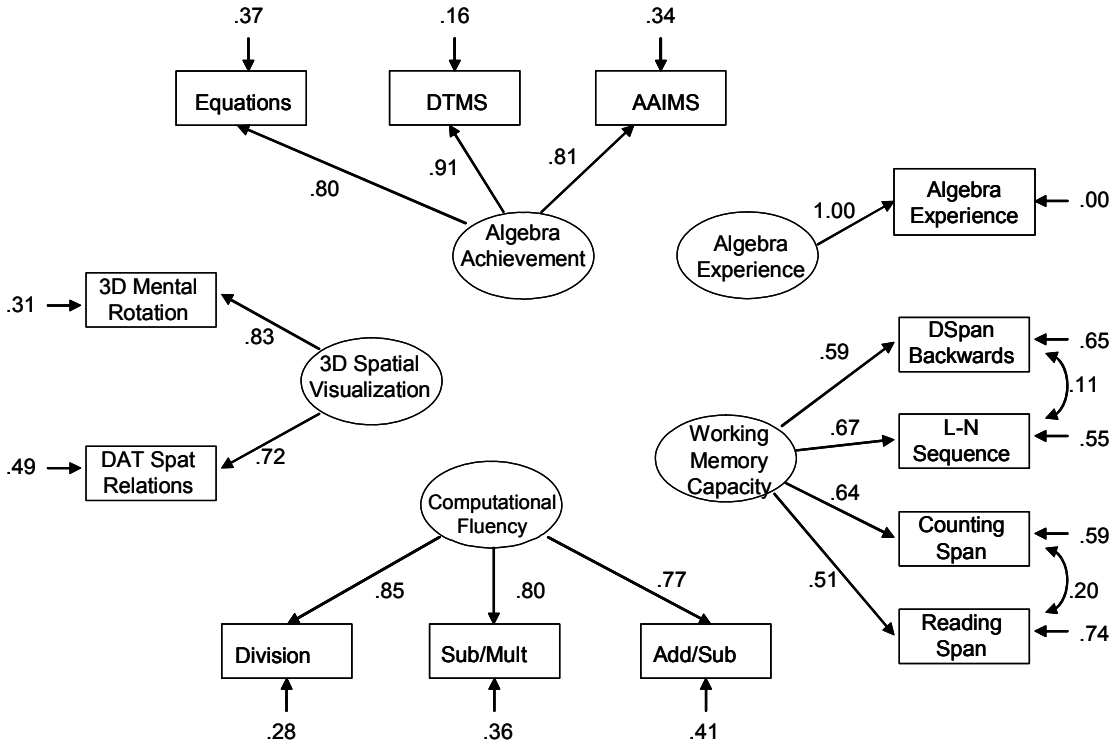


Figure 6. Final measurement model. Although the paths are not shown to simplify the figure, all latent variables were allowed to correlate freely with each other. All loadings and error terms were significant ($p < .05$).

Table 4

Correlations between Latent Variables

	Factor	1	2	3	4	5
1.	Algebra Achievement	-				
2.	Algebra Experience	.60	-			
3.	Computational Fluency	.51	.30	-		
4.	3D Spatial Visualization	.40	.28	.24	-	
5.	Working Memory	.22	.24	.48	.30	-

Note. All correlations were significant ($p < .05$).

The initial model fit the data reasonably well, $\chi^2(56) = 117.72$, p= .000, RMSEA (90% CI)=.07 (.05, .09), CFI = .96, SRMR=.05; however, modification indices indicated that model fit could be improved through the addition of several new paths. Two of the recommended changes, allowing the error terms for digit span – backwards and letter-number sequence and for reading and counting span to covary, were implemented. This resulted in a significantly improved model fit, $\Delta\chi^2 = 17.63$, df = 2, p = .000. The fit of the final measurement model was good, $\chi^2 (54) = 100.09$, p = .000, RMSEA (90% CI) =.06 (.04, .08), CFI = .97, SRMR = .05. Because this model was sufficient for the purposes of testing structural relationships between the latent variables and for theoretical reasons, the other recommended changes were not included in the final model (e.g., adding a path between the indicator subtraction/multiplication and the latent variable 3D spatial visualization, as recommended, would make interpretation of relationships between this construct and the others problematic).

The factor loadings in the final measurement model (see Figure 6) were consistent with those found in previous research studies (Conway et al., 2002; Engle et al., 1999; Kane et al., 2004). The factor loadings for working memory were moderate; whereas, those for 3D spatial visualization were relatively high. The correlations between factors (see Table 4) were intriguing because although the correlation between working memory and computational fluency was moderate, correlations between working memory and the other factors were low with the correlation between working memory and algebra achievement being lower than the rest. Furthermore, the correlations between algebra achievement and the other three predictors were moderate, suggesting that any relationship between working memory and algebra achievement is likely mediated by

some combination of algebra experience, 3D spatial visualization, and computational fluency. To determine, if in fact this was the case, and to test the study hypotheses, several structural models were analyzed.

Structural Models. The initial structural model (see Figure 7) included direct causal paths between all the cognitive predictors and algebra achievement and between algebra experience and achievement. In addition, working memory was set as a causal variable for all other predictors of algebra achievement; 3D spatial visualization was set as a causal variable for computational fluency and algebra experience; and computational fluency was set as a causal variable for algebra experience.

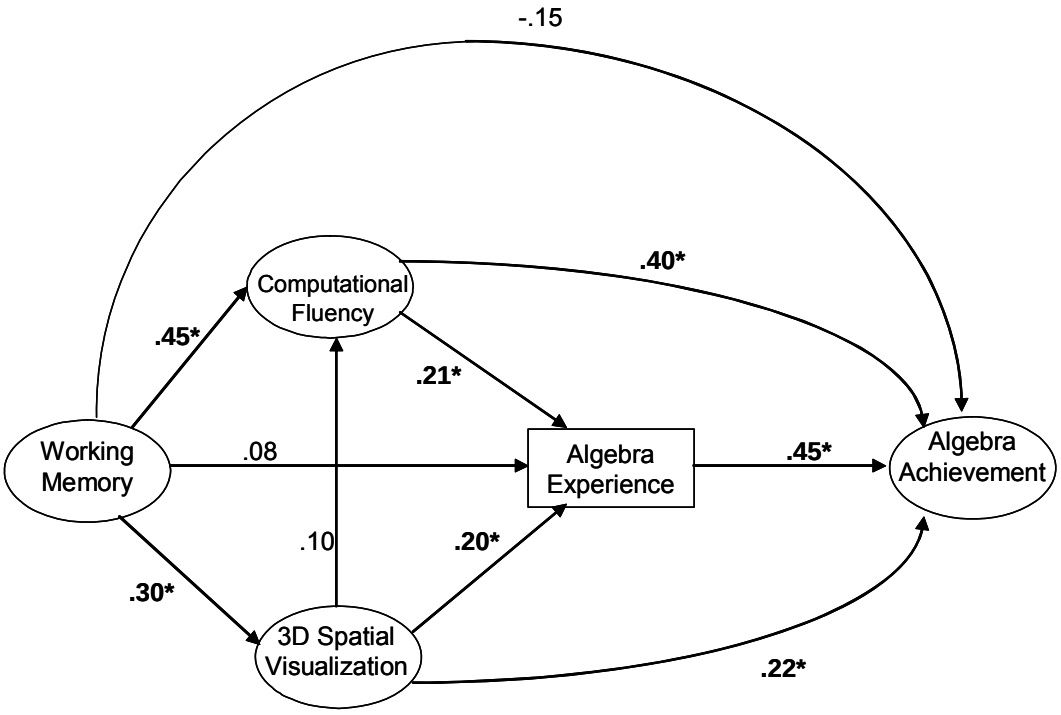


Figure 7. Initial structural model with standardized path coefficients. Starred coefficients were significant ($p < .05$).

The fit statistics for the initial structural model were the same as that of the final measurement model because all the paths between factors were the same even if the implied causal relationships were different. This, of course, meant the model fit was good; however, of more interest was the path coefficients and what they indicated about the relationships between constructs (see Figure 7).

According to the first hypothesis, working memory should have a direct effect on algebra achievement even when controlling for other factors. However, the coefficient for the direct path between working memory and algebra achievement was not significant suggesting that removing this path would not affect model fit, and in fact, when this path was removed, there was no significant change in model fit (see Table 5).

Table 5

Chi-square Differences between Structural Models

Structural Model	χ^2 (df)	$\Delta\chi2$ (df)	p
Initial	100.09 (54)		
H1: Remove Working Memory --> Algebra Achievement	102.78 (55)	2.69 (1)	**0.101**
H2: Remove Working Memory --> Algebra Experience	100.62 (55)	0.53 (1)	**0.467**
Remove Working Memory --> Computational fluency	119.90 (55)	19.81 (1)	0.000
Remove Working Memory --> 3D Spatial Visualization	107.22 (55)	7.13 (1)	0.008
H3: Remove Computational fluency --> Algebra Achievement	124.28 (55)	24.19 (1)	0.000
H4: Remove 3D Spatial Visualization --> Computational fluency	101.38 (55)	1.29 (1)	**0.256**
Remove 3D Spatial Visualization --> Algebra Achievement	109.28 (55)	9.19 (1)	0.002
Remove 3D Spatial Visualization --> Algebra Experience	104.16 (55)	4.07 (1)	0.044
Final	104.84 (57)	4.75 (3)	**0.191**

Note. H = hypothesis. Bolded items indicate paths that could be removed without significantly reducing model fit.

Based on this result, the first hypothesis was not supported. Table 5 shows the effects on model fit when testing the remaining hypotheses through removing specific paths. The

second hypothesis, that working memory directly affects computational fluency, 3D spatial visualization, and algebra experience was only partially supported. Although the paths between working memory and computational fluency and between working memory and 3D spatial visualization were significant and removing them reduced model fit, the path between working memory and algebra experience was not significant and removing it did not significantly affect model fit. According to the third hypothesis, algebra achievement is directly affected by computational fluency and 3D spatial visualization. Consistent with this hypothesis, the direct paths between these constructs were significant and removing them decreased model fit. Finally, the fourth hypothesis that 3D spatial visualization does not directly affect either computational fluency or algebra achievement was only partially supported. The path between 3D spatial visualization and computational fluency was not significant and removing it did not affect model fit; however, the path between 3D spatial visualization and algebra achievement was significant and removing it decreased model fit. The fit of the final model (see Figure 8) with all nonsignificant paths removed was excellent, $\chi^2(57) = 104.84$, $p = .000$, RMSEA (90% CI)=.06 (.04, .08), CFI = .97, SRMR=.05 and not significantly worse than the fit of the initial model. This model accounted for half of the variance in algebra achievement. The combined direct and indirect effects of working memory, 3D spatial visualization, and computational fluency on algebra achievement were .31, .31, and .45, respectively.

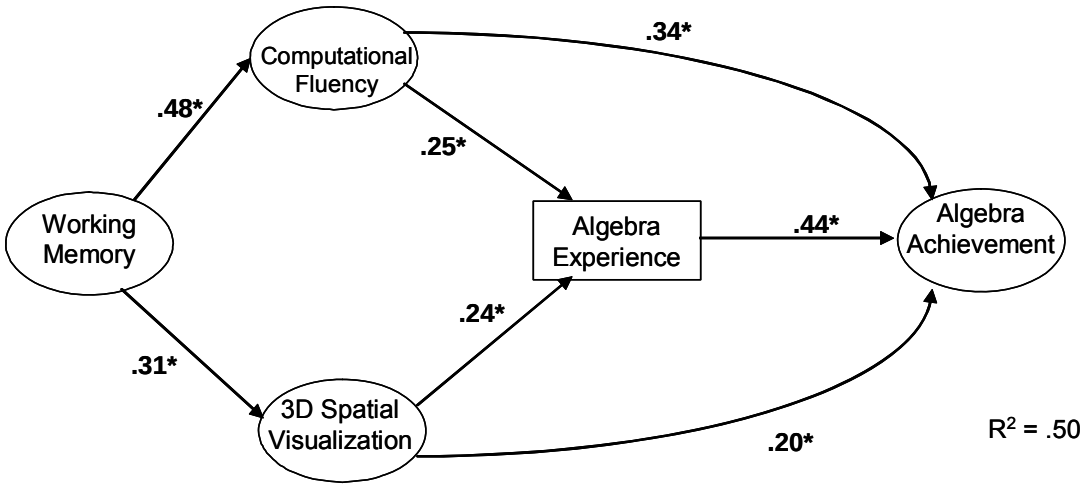


Figure 8. Final structural model. All paths were significant ($p < .05$).

Because algebra experience had a relatively strong relationship with achievement, the effect of experience on achievement was examined in more detail. An ANOVA was run on factors scores for algebra achievement with algebra experience as the group variable. Algebra experience was collapsed from six to three groups: participants who had only taken algebra courses (N = 27), those who had taken advanced algebra/trigonometry or precalculus courses (N = 100), and those who had taken calculus courses (N = 68). There was a significant effect of experience, $F(2, 192) = 46.80$, $p = .000$. Tukey post hoc analysis indicated that students who had taken calculus performed significantly better than those who had not taken calculus with no difference in performances between students who had only had algebra courses and those who had taken trigonometry or precalculus courses. Participants who had taken calculus correctly solved an average of 44 out of 60 problems, whereas participants who had taken only

algebra courses correctly solved an average of 29 problems, and those who had taken trigonometry or precalculus courses correctly solved an average of 32 problems.

Structural Analyses II: Comparison of Algebra Achievement to SAT-Math

To examine whether or not the final cognitive model from the previous analyses is unique to algebra achievement as compared to more general outcomes, models with SAT-M as the outcome were tested. Models with SAT-V as the outcome were also tested to determine if the factors predicting the math constructs were unique to math or more indicative of domain general abilities. These analyses were based on a subsample (N = 135) of the larger group of participants, those who granted permission to access SAT scores. For this group of students, mean SAT-M was 539 (SD = 79, Range: 390 to 800) and mean SAT-V was 530 (SD = 64, Range: 400 to 770). These scores were comparable to the 2005 scores for all college-bound seniors (N = 1,475,623; SAT-V Mean 508, SD 113; SAT-M Mean 520, SD 115; The College Board, 2005), although the variance appeared to be lower among the study participants than the population in general. As Table 6 indicates, students who provided SAT scores did not appear to differ substantially from those who did not provide SAT scores on demographics or on the measures used in this study. The pattern of correlations between measures within factors for the reduced sample (see shaded areas of Table 7) also appeared to be similar to that of the larger group.

Table 6

Comparisons between SAT and non SAT Groups

Demographic/Measure	SAT (N = 135)	No SAT (N = 60)
	Percent	
Females	48	53
Males	52	47
Algebra-based course level		
One algebra course	0	2
Two algebra courses	2	2
Three or more algebra courses	10	14
One or more pre-calculus or advanced algebra/trigonometry course	55	43
Calculus I	30	28
Calculus II or higher	4	10
	Mean (SD)	
Age	19.4 (1.4)	19.8 (1.7)
AAMES Algebra Test	9.9 (3.2)	10.2 (3.2)
Algebra Equations Test	6.3 (2.3)	6.7 (2.6)
DTMS Algebra Subtests	19.2 (6.1)	19.8 (6.5)
DAT Spatial Relations	18.5 (6.2)	18.5 (6.1)
3D Mental Rotation	8.7 (4.5)	9.8 (5.2)
Subtraction/Multiplication Calculations	19.1 (6.4)	19.6 (6.9)
Division Calculations	11.7 (5.3)	11.8 (5.7)
Addition/Subtraction Correction	31.0 (8.7)	31.2 (8.9)
Digit Span- Backwards	7.5 (2.2)	8.0 (2.1)
Letter-Number Sequence	12.1 (2.7)	12.0 (2.4)
Counting Span	34.3 (9.9)	34.7 (10.2)
Reading Span	24.2 (6.5)	24.9 (6.7)

Note. There were no significant differences between groups across measures ($p > .05$).

Table 7

Correlations between Observed Variables among SAT Participants

	Variable	1	2	3	4	5	6	7	8	9	10	11	12	13
1.	SAT-M	-												
2.	SAT-V	***.53***	-											
3.	AAIMS	***.56***	***.17***	-										
4.	AlgebraEq	***.54***	.07	***.63***	-									
5.	DTMS	***.59***	.15	***.73***	***.71***	-								
6.	DAT	***.45***	***.39***	***.27***	.12	***.22***	-							
7.	MRT	***.48***	***.38***	***.25***	***.19***	***.25***	***.65***	-						
8.	Sub/Mult	***.33***	.06	***.33***	***.28***	***.27***	.04	-.08	-					
9.	Division	***.39***	.13	***.34***	***.33***	***.33***	.06	.15	***.65***	-				
10.	Add/Sub	***.45***	***.21***	***.24***	***.24***	***.22***	.11	***.24***	***.60***	***.64***	-			
11.	DSpanB	***.33***	.15	.16	***.12***	***.18***	.11	.07	***.17***	***.27***	***.18***	-		
12.	LNSeq	***.25***	***.27***	.04	.09	.12	.15	***.21***	.12	***.25***	***.32***	***.46***	-	
13.	CSpan	***.20***	.09	.08	.03	.09	.06	.14	***.30***	***.33***	***.31***	***.31***	***.39***	-
14.	RSpan	***.26***	.16	.05	.12	.04	.08	.08	***.20***	***.24***	***.20***	***.23***	***.34***	***.53***

Note. AAIMS = AAIMs algebra test; AlgebraEq = Algebra equations test; DTMS = DTMS elementary and intermediate algebra subtests; DAT = spatial relations test; MRT = 3D mental rotation test; Sub/Mult = subtraction/multiplication computational fluency test; Division = division computational fluency test; Add/Sub = addition/subtraction correction test; DSpanB = Digit span backwards; LNSeq = Letter-number sequence; CSpan = counting span; RSpan = reading span.

The first two columns of Table 7 show the correlations between SAT-M, SAT-V and the other measures. The SAT-M, unlike the algebra measures, correlated with all other observed variables including all of the working memory measures. The SAT-V did not correlate with most of the math or working memory measures, although it did correlate with the spatial visualization measures. Three sets of models were examined,

one each for algebra achievement, SAT-M, SAT-V, to determine if the pattern of relationships between the cognitive factors and these achievement outcomes are likely to differ.

Measurement models. The same measurement model that was used for the larger sample, including correlated error terms for backwards digit span and letter-number sequence and for reading and counting span, was analyzed for algebra achievement with the smaller sample as well as for SAT-M and SAT-V. As Table 8 shows, the fit for the algebra achievement model was excellent. The fits for the SAT models were reasonably good, although based on the RMSEA index, the possibility of poor fits could not be rejected. The fit for the SAT-M model could have been improved substantially ($\Delta\chi^2$ (2) = 25.93) by allowing the subtraction/multiplication and addition/subtraction correction errors to correlate with the 3D mental rotation error, but as already mentioned, interpretation of the final model would have been problematic. For the purposes of the exploratory nature of these analyses, the measurement model fits were considered good enough to continue with examination of structural models.

Table 8

Fit Indices for Cognitive Models of Achievement

Model	df	χ^2	p	RMSEA (90% CI)	CFI	SRMR
Algebra Measurement	54	69.95	.07	.04 (.00, .07)	.98	.04
SAT-M Measurement	34	61.44	.00	.07 (.04, .10)	.96	.05
SAT-V Measurement	34	49.98	.04	.06 (.00, .09)	.97	.05
Algebra Structural	57	72.65	.08	.04 (.00, .07)	.98	.05
SAT-M Structural	37	62.17	.01	.06 (.03, .10)	.96	.05
SAT-V Structural	37	52.42	.05	.05 (.00, .09)	.97	.05

Structural models. Full structural models (see Figure 7 for example) including direct causal paths from working memory to the achievement variables were tested, and in all cases, working memory was not a significant predictor of achievement when controlling for other factors. Paths connecting working memory to achievement, working memory to algebra experience, and 3D spatial visualization to computational fluency were removed to replicate the model trimming procedures implemented in the first set of analyses. For all three achievement models, model fits were not significantly different without these paths (Algebra $\Delta\chi^2$ (3) = 2.70, p = 0.440; SAT-M $\Delta\chi^2$ (3) = 0.73, p = 0.866; SAT-V, $\Delta\chi^2$ (3) = 2.44, p = 0.486). These steps resulted in the models shown in Figures 9 (a), 9 (b), and 9 (c). Based on the fit indices shown in Table 8, these models fit the data reasonably well.

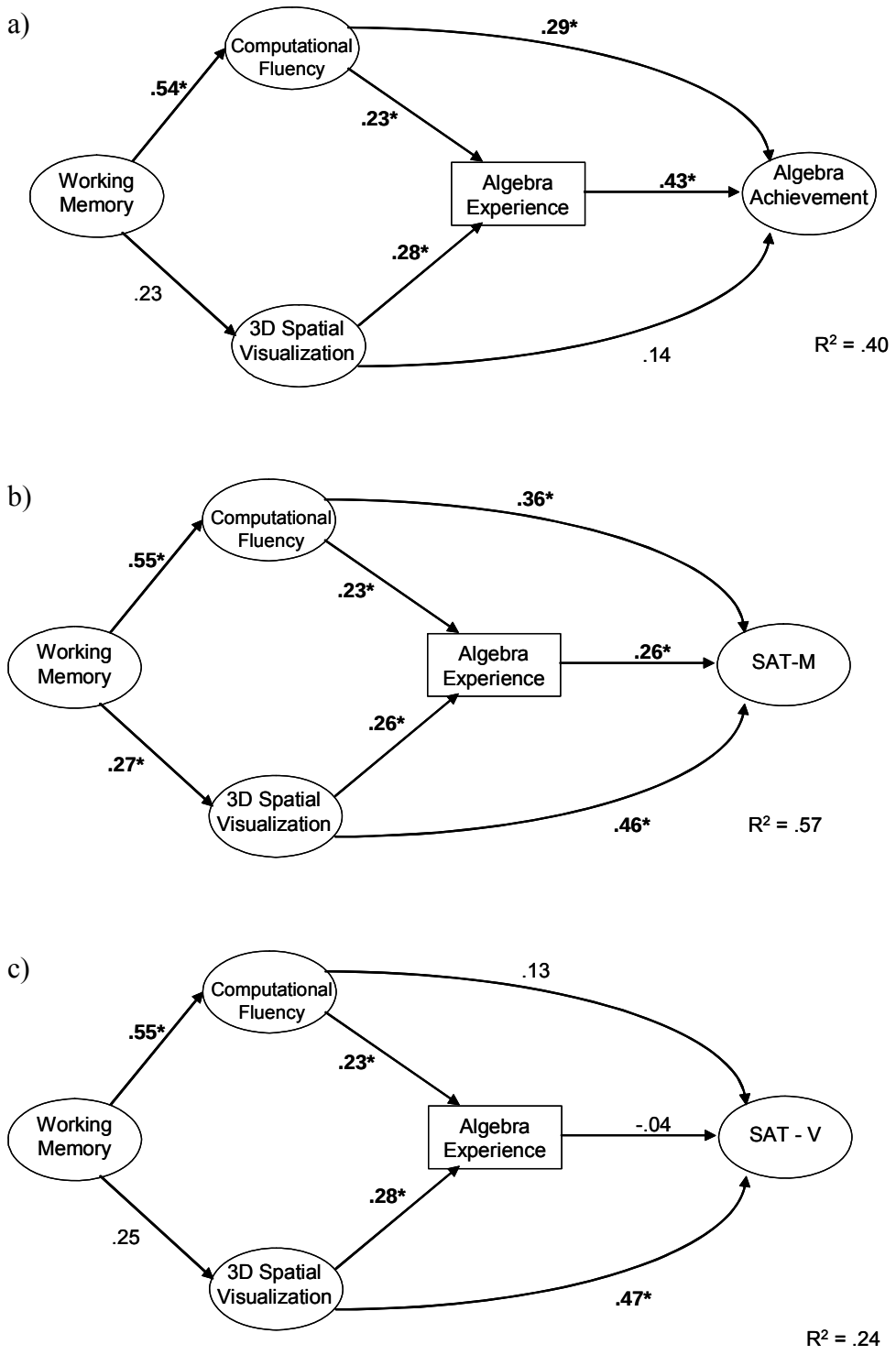


Figure 9. Final models for algebra achievement (a), SAT-M (b), and SAT-V (c). Starred path coefficients were significant ($p < .05$).

In the model for algebra achievement shown in Figure 9 (a), the paths connecting working memory to 3D spatial visualization and 3D spatial visualization to algebra achievement were not significant. This result is different than what was found in the larger sample and could be a function of sample size. The relative strength of the direct effects, however, was the same for both models. In addition, the pattern of total effects were similar to those found in the larger sample. The total effects of working memory, 3D spatial visualization, and computational fluency on algebra achievement in the smaller sample were .27, .26, and .39, respectively. This contrasts markedly, though, with the relative strengths of effects on SAT-M. Based on the magnitude of the standardized path coefficients, algebra experience had the largest direct effect on algebra achievement followed by computational fluency, then 3D spatial visualization. The pattern of direct effects was completely reversed for SAT-M with 3D spatial visualization having the strongest effect, followed by computational fluency then algebra experience. The pattern of total effects was also different. The total effects for working memory, 3D spatial visualization, and computational fluency were .37, .53, and .42, respectively. The effect of 3D spatial visualization on SAT-V was similar in both relative and absolute strength to its effect on SAT-M. However, computational fluency and algebra experience had no effect on SAT-V supporting the hypothesis that the effects they have on math achievement are domain specific.

Structural Analyses III: Examination of the influence of cognitive factors on algebra achievement by gender.

The goal of this next set of analyses was to examine cognitive models of algebra achievement for potential gender differences in the pattern of relationships. All analyses were conducted on the entire sample (N = 195). Tables 1 and 9 show that the males had higher levels of math experience than the females and outperformed the females on most of the measures except for 3 of the 4 working memory tasks, one of the algebra tests, and one of the computational fluency tasks.

Table 9

Mean (SD) Performance by Gender

Measure	Females (N = 97)	Males (N = 98)	t	p
AAMES Algebra Test	9.8 (3.1)	10.2 (3.3)	0.94	0.348
Algebra Equations Test	6.0 (2.2)	6.9 (2.4)	2.77	0.006
DTMS Algebra Subtests	18.0 (5.9)	20.8 (6.2)	3.22	0.001
DAT Spatial Relations	17.0 (5.6)	19.7 (6.5)	2.77	0.006
3D Mental Rotation	7.2 (3.7)	11.0 (4.9)	6.11	0.000
Subtraction/Multiplication	18.8 (5.8)	20.0 (7.2)	0.87	0.383
Division	10.6 (4.6)	13.0 (5.9)	2.84	0.005
Addition/Subtraction	28.1 (5.8)	33.9 (10.1)	4.90	0.000
Digit Span- Backwards	7.6 (2.1)	7.8 (2.2)	0.77	0.441
Letter-Number Sequence	11.5 (2.5)	12.7 (2.7)	3.19	0.002
Counting Span	33.6 (9.3)	35.2 (10.6)	1.10	0.272
Reading Span	24.3 (6.4)	24.6 (6.7)	0.28	0.780

The most marked difference in performance between males and females was on the 3D Mental rotation task with well over a half of a standard deviation between average performances of males and females. More germane to the goal of these analyses, though, is the comparison of the patterns of correlations between measures shown in Table 10.

Correlations between measures within the 3D spatial visualization and computational fluency factors appeared to be higher for males than females whereas correlations between the measures within the algebra and working memory constructs were similar across gender. In addition, all correlations between the 3D spatial visualization measures and the algebra measures were significant for the males whereas all but one of these correlations was not significant for the females. There was a similar pattern in the correlations between the computational fluency and algebra measure; however, about half of these correlations were significant for the females. One counterintuitive result for the females was a significant negative correlation between subtraction/multiplication and 3D mental rotation. The one similarity between males and females was in the pattern of correlations between algebra and working memory measures.

A series of SEM models were tested to determine what these patterns meant in terms of structural relationships between factors across gender. To determine if there are gender differences, models in which parameter are constrained to be equal across gender must be compared to models in which the parameters are allowed to differ across gender. Significant differences in model fit provide support for gender differences in the path coefficients being tested.

Table 10

Correlations between Observed Variables by Gender

	Measure	1	2	3	4	5	6	7	8	9	10	11
1.	AAMES Algebra											
2.	Algebra Equations	**.60/ .67**										
3.	DTMS Algebra	**.72/ .79**	**.68/ .74**									
4.	DAT Spatial Relations	.18/ **.45**	.01/ **.25**	.08/ **.35**								
5.	3D Mental Rotation	**.20/ .33**	.09/ **.23**	.14/ **.31**	**.42/ .68**							
6.	Subtraction/ Multiplication	.19/ **.43**	**.21/ .41**	**.20/ .45**	-.13/ .15	**-.25/** .13						
7.	Division	**.36/ .42**	**.29/ .43**	**.31/ .46**	-.07/ .18	-.01/ **.23**	**.56/ .75**					
8.	Addition/ Subtraction	.11/ **.38**	.17/ **.32**	.14/ **.34**	-.19/ **.23**	.03/ **.31**	**.60/ .71**	**.51/ .67**				
9.	Digit Span-Backwards	.09/ .19	.00/ **.22**	.08/ **.26**	.09/ .09	.08/ .17	.08/ **.24**	.13/ **.40**	.05/ **.25**			
10.	Letter-Number Sequence	.02/ .11	.09/ .11	.09/ .14	.06/ .15	.15/ **.22**	.03/ .18	.12/ **.37**	.13/ **.33**	**.61/ .42**		
11.	Counting Span	.13/ .05	.03/ .05	.12/ .03	.11/ -.04	.17/ .12	.15/ **.28**	**.21/ .37**	.10/ **.35**	**.41/ .39**	**.43/ .39**	
12.	Reading Span	.01/ .13	.13/ .11	.04/ .09	.16/ -.05	.09/ .04	.11/ **.24**	.16/ **.31**	.06/ **.21**	**.33/ .26**	**.42/ .30**	**.48/ .57**

Note. Females/Males.

Measurement models. Separate measurement models were examined for males and females to determine if the 12 measures loaded onto the four factors for both males and females. The model fits were good for each group (females: $\chi^2(54) = 60.66$, p= .25, RMSEA (90% CI)=.03 (.00, .08), CFI = .98, SRMR=.06; males: $\chi^2(54) = 57.03$, p= .36, RMSEA (90% CI)=.02 (.00, .07), CFI = 1.00, SRMR=.05). Then a combined

measurement model in which all parameters were allowed to vary across gender was examined. This fit of this model was also good ($\chi^2(108) = 117.7$, p= .25, RMSEA (90% CI)=.03 (.00, .06), CFI = .99, SRMR=.05). However, there were noticeable differences in some of the parameters across gender. For males, the factor loadings for 3D spatial visualization were both high (DAT spatial relations = .87, 3D MRT = .78) whereas for females, while 3D MRT loaded highly (.95), DAT loaded only moderately (.44) onto 3D spatial visualization. In addition, the difference between genders in the pattern of correlations between factors was striking (see Table 11).

Table 11

Correlations between Latent Variables by Gender

	Factor	1	2	3	4	5
1.	Algebra Achievement	-				
2.	Algebra Experience	**.41/ .70**	-			
3.	Computational Fluency	**.34/ .56**	.13/ **.33**	-		
4.	3D Spatial Visualization	.19/ **.45**	-.04/ **.29**	-.17/ **.27**	-	
5.	Working Memory	.13/ .22	**.30**/ .13	.23/ **.60**	.20/ .16	-

Note. Females/Males

Algebra achievement did not correlate with working memory and did correlate with algebra experience and computational fluency for both groups. All other correlations appeared to differ by gender. Algebra experience correlated with working memory, but

not with computational fluency or 3D spatial visualization for the females, but the pattern of correlations was opposite for the males. Also, computational fluency correlated with 3D spatial visualization and working memory for the males, but did not correlate for the females. The results of this measurement model, however, do not provide definitive evidence of gender differences either in relations between measures and constructs or in relations between constructs.

A second measurement model was tested in which all factor loadings and error terms were constrained to be equal across gender. Variances and covariances between the latent variables were allowed to differ across gender in order to determine the best fitting measurement model across the two groups before testing a structural model. There was no difference in fit between this model and the model in which all parameters were allowed to differ across gender, $\Delta\chi^2$ (22) = 32.72, p = .07 (overall model fit: χ^2(130) = 150.42, p= .11, RMSEA (90% CI)=.03 (.00, .06), CFI = .98, SRMR=.07). It should be noted that the difference in chi-square was close to being significant and perhaps with larger samples of males and females, the results would be different given the difference in loadings for spatial visualization when the groups were tested independently. However, because in this case the difference was not significant and for the sake of parsimony, a measurement model in which all factor loadings and error terms were constrained to be equal across gender was used for subsequent structural analyses.

Structural Models. The full structural model in which all causal paths between the cognitive factors and algebra achievement were included was tested first (see Figure 10).

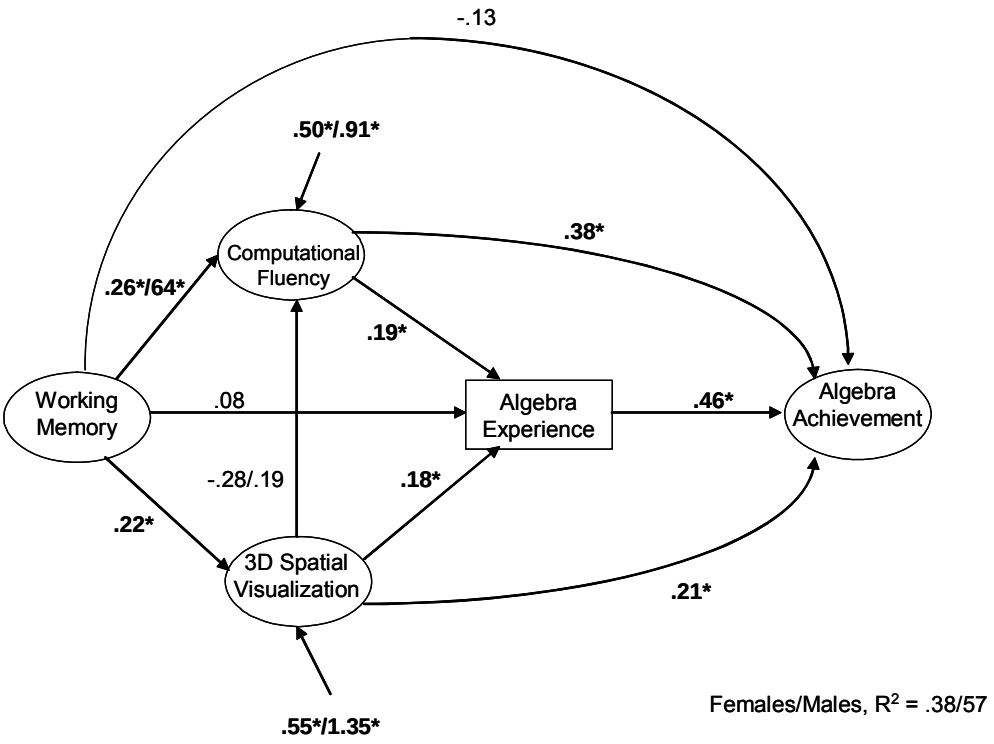


Figure 10. Initial structural model of algebra achievement by gender.

The fit of a model in which all path coefficients were constrained to be equal across gender was significantly worse than one in which all coefficients were allowed to vary across gender, $\Delta\chi^2$ (15) = 44.19, p = .000 (fully-constrained model fit: $\chi^2(145)$ = 194.61, p= .00, RMSEA (90% CI)=.05 (.03, .08), CFI = .96, SRMR=.15). Based on modification indices, structural paths and factor variances were freed to vary across gender, one parameter at a time. The results of these changes are shown in Table 12 and Figure 10. The fit of the resulting model was quite good, $\chi^2(141)$ = 163.87, p= .09, RMSEA (90% CI)=.04 (.00, .07), CFI = .98, SRMR=.07)

Table 12

Chi-square Differences between Gender Structural Models

Structural Model	df	χ2	df	Δχ2	p
All paths constrained to be equal across gender	145	194.61			
Free path from working memory to computational fluency	144	181.78	1	12.83	0.000
Free variance in 3D spatial visualization	143	172.70	1	9.08	0.003
Free path from 3D spatial visualization to computational fluency	142	167.76	1	4.94	0.026
Free variance in computational fluency	141	163.87	1	3.89	0.049

Finally, removing the non-significant paths between working memory and algebra achievement, working memory and algebra experience, and 3D spatial visualization and computational fluency to reflect the final model from the first set of structural analyses did not affect the model fit, $\Delta\chi^2$ (4) = 9.22, p = .056 (final model fit: $\chi^2(145)$ = 173.09, p= .045, RMSEA (90% CI)=.04 (.03, .07), CFI = .98, SRMR=.08). Figure 11 shows the results of these changes.

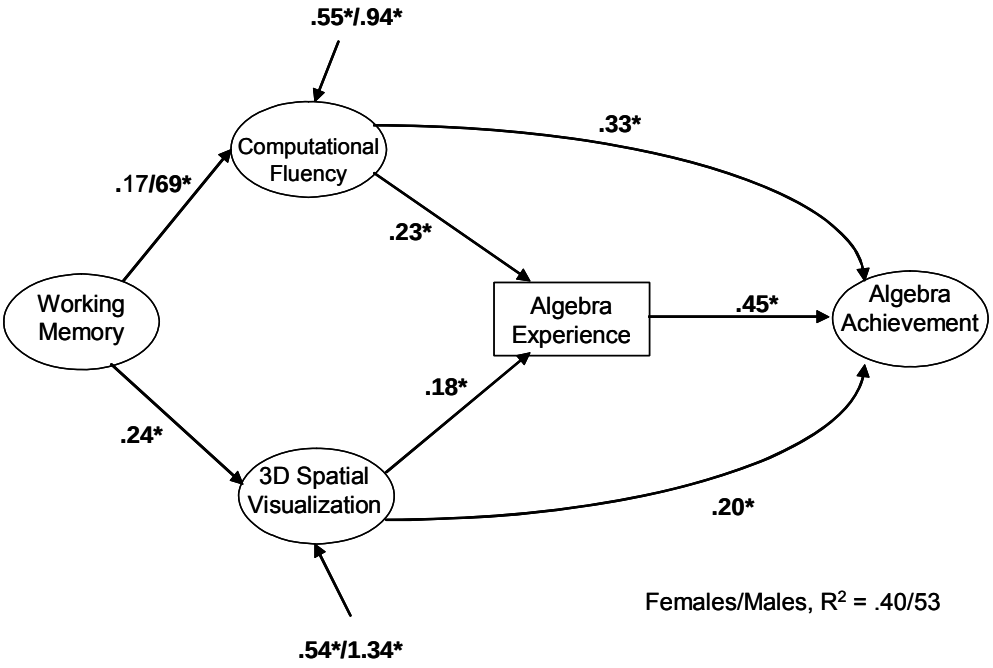


Figure 11. Final model of algebra achievement by gender.

This model appears to account for more variance in algebra achievement among males than among females. The coefficients for the direct effects of algebra experience, computational fluency, and 3D spatial visualization on algebra achievement did not differ by gender. However, the strength of the effect of working memory on computational fluency did differ by gender. In fact, this path coefficient was high for males and nonsignificant for females.

The unexplained variance in computational fluency and 3D spatial visualization is greater for males than females; however, it should be noted that the unexplained variance in 3D spatial visualization for males is greater than one. This is an impossible result because these values are standardized and a disturbance value greater than one would mean that more than 100% of the variance is unexplained. In addition to this result being impossible, the pattern of disturbance differences in the multi-group model is not consistent with the pattern of disturbances across the separate structural models that were analyzed for males and females. In these separate models, the unexplained variance in 3D spatial visualization was similar for males and females (.97 vs. .90). The unexplained variance for computational fluency was highly dissimilar across gender (.61 for males vs. .90 for females) in the separate models, but in the opposite direction than was the case in the multi-group model. The pattern in the separate models is more consistent with the finding that working memory explains more variance in computational fluency for males than for females.

These contradictory results between the separate models for each gender and the single multi-group model as well as the impossible result in the multi-group model are

likely a function of several factors. There were large differences in the variances on the spatial visualization and computational fluency measures across gender. The variance in performance was much higher among males than females (from 32% more variance in the DAT Spatial Relations task to 300% more variance in the addition/subtraction correction task). In addition, the pattern of correlations between factors was quite different for males and females. The relatively small sample sizes within groups are likely inadequate for analyzing these extreme group differences within the imposed measurement and structural constraints in the multi-group model.

Discussion

Algebra achievement appears to be largely a function of domain specific factors. Although domain general abilities are also important, their impacts on algebra achievement seem to be primarily mediated through more domain specific factors. The results of this study suggest that algebra experience and computational fluency are each highly related to algebra achievement, more so than either 3D spatial visualization or working memory. In addition, the more dominant role of algebra experience and computational fluency as compared to more domain general abilities seems to be unique to algebra achievement when compared to broader measures of math achievement such as the SAT-M.

The main hypothesis of this study was that working memory would have a direct effect on algebra achievement when controlling for other factors. The absence of a direct effect of working memory on algebra achievement clearly means this hypothesis was not supported. The original hypothesis was partially based on the overwhelming evidence that working memory is related to math achievement among children (e.g., Floyd et al.,

2003; e.g., Swanson & Jerman, 2006) and with only two exceptions (Reuhkala, 2001; Rohde & Thompson, 2007), working memory is related to math achievement among adolescents and adults (e.g., Engle et al., 1999; e.g., Floyd et al., 2003). One of the suggested reasons for the inconsistencies among studies of adolescents and adults was that differences in the measures of working memory across studies were causing differences in the strength of the effects. The evidence from this study does not support this hypothesis.

In this study, working memory was based on multiple measures, each administered individually. This method of representing working memory as well as the moderate correlations between the working memory measures are consistent with studies in which working memory is assumed to be constrained by domain general executive attention and has been found to be a robust predictor of abstract reasoning among adults (Conway et al., 2002; Engle et al., 1999; Kane et al., 2004). Furthermore, the strength of the relation between working memory and SAT-M scores in this study is almost identical to the one found between working memory and SAT-M scores in a study which was based on similar methodologies for measuring working memory (Engle et al., 1999). The low correlation between algebra achievement and working memory is also consistent with evidence that working memory is not related to higher level math achievement among adolescents (Reuhkala, 2001). In the Reuhkala (2001) study, although the correlation between verbal working memory and math achievement was not significant (and based on a much smaller sample size than the one used in this study), the size of the correlation coefficient was similar to the size of the correlation found between working memory and algebra achievement. Although in this study and the Reuhkala study, the

methodologies for measuring working memory were different, the assessment of higher level math achievement among adolescents and algebra achievement among adults were similar in that they emphasized acquired procedural knowledge over more fluid mathematical problem solving abilities. The results from this study in conjunction with the pattern of results from other studies suggest that it is not differences in measures of working memory that are causing the different outcomes, but differences in how math achievement is defined. Finally, the results from this study provide a more nuanced explanation for the effect of working memory on higher level math achievement than was suggested by evidence that working memory is not related to higher level math achievement when controlling for other factors such as processing speed and spatial abilities (Rohde & Thompson, 2007). Among adolescents and adults, it appears that working memory does influence higher level math achievement, but the influence is mediated by other cognitive factors. The evidence form this study combined with the evidence from the Rohde and Thompson (2007) study suggest that three of these factors are computational fluency, spatial processing abilities, and processing speed.

Another hypothesis of this study was that 3D spatial visualization would not have a direct effect on algebra achievement. This hypothesis was not supported by the evidence. Surprisingly, the strength of the correlation between 3D spatial visualization and algebra achievement is similar to the correlations between 3D mental rotation and higher level math achievement among adolescents found in two other studies (Reuhkala, 2001). Although the math achievement measures were similar across studies in that they emphasized procedural knowledge, they differed in a critical way in terms of the expected influences of visual-spatial abilities. The math assessments in the Reuhkala

(2001) studies included geometry problems which is one possible reason why visual-spatial working memory, but not verbal working memory was related to math achievement. None of the problems in the algebra achievement tests specifically required processing of 3D spatial information. Only 10% of the problems required processing of 2D graphs or number lines.

One possible explanation for the unexpected effect of 3D spatial visualization on algebra achievement is that the direct effect of 3D spatial visualization on algebra achievement is primarily due to domain general processes. This hypothesis is consistent with evidence of the strong relationship that 3D spatial visualization has with domain general executive processes such as controlled attention (Miyake et al., 2001). The evidence from this study that verbal working memory has a direct effect on 3D spatial visualization and that 3D spatial visualization has a direct effect on verbal achievement also implicates domain general processes. It is not clear, though, what domain general processes are most influencing these effects. It is possible that 3D spatial visualization involves some domain general processes associated with working memory, processes which have similar effects on lower level math skills such as computational fluency. This conjecture is consistent with the result that 3D spatial visualization is not related to computational fluency when controlling for working memory. It is also possible that 3D spatial visualization involves some domain general process that are not involved in working memory and that these processes have a stronger direct effect on higher level math achievement than those associated with working memory, although the total effects of 3D spatial visualization and working memory on higher level math achievement are similar. The evidence from this study is consistent with, but does not provide definitive

support for these hypotheses. The evidence from this study does suggest that explicating the processes involved in 3D spatial visualization and the way in which each of these processes influence higher level math achievement is an area of research that needs more attention.

It was expected that domain general processes would play a stronger role in algebra achievement than domain specific processes. However, the direct effect of algebra experience and the total effects of computational fluency were much higher than the total effects of working memory and 3D spatial visualization. Part of the explanation for the dominant role of algebra experience and computational fluency is likely due to the way in which algebra achievement is defined. In this study, algebra achievement was defined as symbol manipulation. This is a narrow definition of algebra achievement because it does not include representing and solving word problems, pattern recognition, modeling, and other abilities that have been suggested as key factors in algebraic thinking (Bell, 1996; Janvier, 1996; Mason, 1996; Rojano, 1996). However, the problems used in this study are typical of the type of problems found in many high school and college algebra curriculums and involved solving algebraic equations, simplifying algebraic expressions, and translating between symbolic and graphical representations of functional relations. Solving these types of problems involves retrieving textbook procedures and algorithms from long term memory and implementing them. If a procedure is not readily available in memory, a low or average performer may stop without trying to use mathematical reasoning or arithmetical means to solve the problem (Goodson-Espy, 1998). The behavior of the participants in this study is consistent with that approach because the vast majority of participants stopped working on the algebra tests before they

were required to stop. On the algebra equations open response test, problems were often left blank without any apparent attempt to solve them in spite of the fact that some of them could have been solved by trial and error with single digit numbers (e.g., $x = x^2$). For simpler problems (e.g., $2x + 3 = 5$), though, the low or average performer may try guess-and-check (i.e., substitute 1 for x and see if this produces a correct solution, if not, try 2, etc.) or undoing (i.e., undo the addition of 3 by subtracting it from 5, etc.; Kieran, 1990, 1992).

High level performers are also likely to retrieve procedures and algorithms from long term memory, although they are likely to have a wider variety of algorithms available to them and be more fluent in implementing them. High performers are also likely to try other arithmetical means for solving the problems if a procedure is not readily available (Goodson-Espy, 1998). They may also use estimation and calculation to monitor progress and to determine if the final answer is correct, or on multiple choice problems, to test the options to determine which option is correct. Finally, high performers may also be flexible enough in their procedural knowledge to appreciate inverse relationships and use this knowledge to determine the answer when choices are available to them. For example, given a factoring problem such as $36x^2 – 1 =$, a high performer may factor the problem as intended, or if the algorithm for factoring is not easily retrieved from memory, may recognize that factoring is the inverse of multiplication and determine that answer using a multiplication algorithm on the suggested answers (e.g., $(6x-1)(6x+1)$), then choosing the one that matches the initial problem. Regardless of the level of performer, the emphasis is on procedural knowledge that is not intuitive and often implemented without appreciation for the concepts that

underlie them or the structure of the problem. The ‘delicate shift of attention’ as suggested by Mason (1989) is not necessarily part of the process in solving these problems. That is not to say that they could not be, but on this type of assessment, students are likely to perform almost exclusively at the procedural level, at least partly because that is the way they are taught and because they are reinforced for doing so (Sjostrom, 2000). This emphasis on procedure, both at the algebraic and numerical levels, may be why algebra experience and computational fluency are more influential on algebra achievement than is working memory and 3D spatial visualization.

Algebra experience has the strongest direct effect on algebra achievement. Fluency in algebra procedural knowledge appears to take years to develop and the key to true fluency may be experience at the calculus level of mathematics. In this study, participants who had taken calculus correctly solved almost 50% more problems on average than those who had not taken calculus. This type of performance is consistent with evidence that suggests that the typical student is unlikely to develop structural awareness of algebraic representations until they have experienced calculus, although even at this level, many students do not have a structural understanding of algebra (English & Sharry, 1996). However, at the calculus level, students appear to at least be relatively fluent in procedural knowledge. Perhaps this is due to the way in which algebra is treated in calculus as opposed to lower level math courses. At the calculus level, algebra is a tool and no longer the focus of study. Practice in algebraic manipulations in a calculus class is an organic part of a more complex problem solving process whereas practice in lower level algebra courses is often rote, repetitive, and divorced from any meaningful mathematical context. Perhaps this different way of experiencing algebra is

critical to both the development of fluency in symbolic manipulations and structural awareness of what those symbols represent.

The role of algebra experience in predicting algebra achievement is not solely a function of actual experience in the classroom. The evidence from this study suggests that students who take higher level math also have more skill in computational fluency as well as better 3D spatial processing abilities and more working memory capacity. High school mathematics is considered by many to be a sieve which sifts out students without the resources necessary to climb the educational ladder. The percentages of students in college preparatory mathematics courses decline dramatically from the first year algebra class to precalculus (U.S. Department of Education, 1997). As a result, a minority of students experience calculus level mathematics. The resources that determine which students are a part of this select group is a subject of much debate and include social and political factors as well as cognitive abilities. Although some of those cognitive factors have been identified in this study, they account for a relatively small amount of the variance in algebra experience. In addition, it cannot be concluded from this study that the relationship between these cognitive abilities and experience level is solely or even primarily a causal one. Although fluency in numerical calculations may increase the likelihood of a student taking higher level math, it is also likely that experience in higher level math improves computational fluency.

The combined direct and indirect effects of computational fluency were as strong as the direct effect of algebra experience on algebra achievement. There are several possible reasons for this which includes both prima facie similarities between the computational fluency tasks and the algebra achievement assessments as well as

underlying causes that are less obvious. Similar to algebra achievement, computational fluency was defined somewhat narrowly for this study. The emphasis in the computational fluency tasks was on rapid multi-digit calculations which include retrieval of single-digit math facts from long term memory as well as retrieval and implementation of computational algorithms. At least part of the relationship between computational fluency and algebra achievement may be due to the ability to store, retrieve, and implement mathematical procedures independent of conceptual understanding. However, although speed was essential in the computational fluency tasks, it was not a manifest requirement in the algebra tasks. So, mathematical procedural efficiency is unlikely the only explanation for the relationship between computational fluency and algebra achievement.

Both low and high performers were likely to have relied primarily on procedural knowledge on the computational fluency tasks; however, high performers were probably more likely to retrieve single-digit and some multi-digit math facts directly from memory. In addition, high performers were probably more likely to use alternative methods to solve the problems such as estimation and algorithms not traditionally taught in the classroom. For example, in the addition/subtraction correction problems, computing the solution to a double digit problem then comparing it to the suggested answer would not necessarily be the most efficient method. A high performer is likely to quickly recognize that problems such as 21 – 10 = 21 or 10 + 27 = 27, are impossible relations whereas a low performer may automatically do the computation without examining the problem as a whole. Flexible and conceptually based numerical fluency

would increase the likelihood of high performers using estimation and other methods suggested earlier on the algebra achievement tests.

The behavior of the participants in this study was consistent with the suggestion that low performers do the problems step-by-step whereas high performers examine them in a more holistic way. For example, low performers were more likely than high performers to show obvious signs of using standard procedures (e.g., rewrite 316 ÷ 4 in the more traditional long division format and solve the problem using that algorithm). However, it is difficult to determine from this anecdotal evidence if participants who did not write out the procedural steps in this way were using estimation or non traditional ways of determining the solutions or if they were simply performing traditional procedures mentally.

Whatever methods the participants were using to solve the computational fluency problems, the tendency of the high performers to not rely on paper-and-pencil notations suggests that they had more working memory capacity with which to maintain procedural steps in short term memory or to solve the problems by the flexible use of alternative methods. Evidence from this study is consistent with this hypothesis. Working memory was directly related to computational fluency and this relationship was relatively strong. This result is also consistent with both correlational and experimental research in which working memory has been found to play a direct role in computational abilities among adolescents and adults (DeStefano & LeFevre, 2004; Floyd et al., 2003). Because computational fluency mediated the relationship between working memory and algebra achievement, the role that working memory capacity plays in both computational fluency and algebra achievement is another possible reason for the strong relationship between

computational fluency and algebra achievement. To what extent working memory influences this relationship in comparison to quantitative abilities such as estimation and to procedural abilities cannot be determined from these results, although the absence of a direct effect and the moderate total effect of working memory on algebra achievement suggests that number sense and mathematical procedural abilities are likely the most critical factors in the relationship between computational fluency and algebra achievement.

Although domain general processes do play a role in algebra achievement, they appear to play less of a role in algebra achievement than in other domains of mathematics. The total effects of 3D spatial visualization and working memory on algebra achievement were much less than the total effect of computational fluency and the direct effect of algebra experience. The pattern of effects was completely different for SAT-M. The total effect of 3D spatial visualization on SAT-M was twice the effect of 3D spatial visualization on algebra achievement. Although the total effect of computational fluency on math achievement was similar in the two models, the effect of working memory was higher whereas the effect of algebra experience was substantially lower on SAT-M than on algebra achievement. Part of the explanation for the difference in these cognitive models is that the SAT-M includes geometric reasoning problems whereas algebra achievement does not. This probably has some influence on the difference in the effects of 3D spatial visualization, but given the evidence that 3D spatial visualization is greatly influenced by domain general processes, it seems unlikely that the inclusion of geometry problems entirely accounts for differences in the effects of 3D spatial visualization. Geometry problems would also not account for the difference in

total effects of working memory across the cognitive models. The marked difference in the pattern of effects in the two models may be partly due to the emphasis on textbook procedural knowledge in algebra achievement and the lesser role this type of knowledge has on the SAT-M. Aside from the inclusion of geometric reasoning, SAT-M also requires more novel problem solving abilities. According to one analysis, 58% of the problems on the SAT-M require textbook style algorithms and 40% may be solved more quickly using estimation or insight, require insightful use of textbook algorithms, or require unique algorithms developed for the specific problem (Gallagher, 1992). The "ability to shift from algorithmic to intuitive strategies may be a critical component of SAT-M test performance but is less likely to be an advantage for students on classroom tests . . . In a classroom context, specific solution strategies are expected to be applied to specific types of problems." p. 182 (Gallagher *et al.*, 2000). The algebra assessments used in this study are typical of those used in a classroom context.

The evidence from this study suggests that the type of achievement tests which have profound influence on a student's options after high school differ substantially from the types of assessments used in the classroom not only in terms of problem types, but also in terms of the demands they place on different cognitive resources. In addition, classroom math experience appears to be less of a factor on SAT-M performance than domain general resources. Although this study provides some support for the hypothesis that classroom experience does not promote the kind of problem solving abilities that are a substantial part of the SAT-M, this evidence needs to be treated with extreme caution. In the algebra achievement model, algebra experience was reflective of actual experience prior to the assessment of achievement, however, it is likely that many of the participants

who had taken calculus took the SAT-M before they had completed calculus. As already mentioned, calculus experience seems to have a profound effect on algebra achievement. The type of mathematical thinking required in calculus courses may also be more consistent with the type of problems that require a more fluid understanding of mathematics found on the SAT-M. If this is the case, the relationship between SAT-M performance and algebra experience may be stronger if the SAT-M is taken after calculus and this stronger relationship may be similar to the one found between algebra experience and algebra achievement. A more definitive way to test this conjecture would be to measure both types of math achievement at the same time, and include actual math experience as a control variable.

In addition to implications as to the role of different cognitive processes in algebra achievement and how the pattern of relationships between these processes may be unique to algebra achievement, evidence from this study also has implications concerning gender differences in math achievement. Some researchers have suggested that the differences in the types of problem solving abilities promoted in the classroom from the types of problem solving abilities required for the SAT-M is one reason for the relatively large gender difference favoring males in SAT-M performance in spite of a slight female advantage in math grades at the high school and college levels (e.g., for review, see Royer et al., 1999). There is also evidence that 3D spatial abilities mediate gender differences in SAT-M performance (Casey et al., 1997). In this study, 3D spatial visualization was more highly related to SAT-M performance than it was to algebra achievement. In addition, there was no gender difference in the effects of 3D spatial visualization on algebra achievement. This suggests that 3D spatial visualization plays a

similar role in math achievement for males and females and because 3D spatial visualization is a more important factor in performance on the SAT-M than it is in algebra achievement, gender differences in 3D spatial abilities may have more of an influence on SAT-M performance that it does in algebra achievement.

Computational fluency has received less attention than 3D spatial visualization in gender research related to math achievement. However, the results of this study suggest that it should receive more research attention, particularly the relationship between working memory, computational fluency, and higher level math achievement. Although computational fluency has a strong effect on algebra achievement for both males and females, and working memory is highly related to computational fluency among males, it is not related to computational fluency among females. This result seems counterintuitive and it would be a valid argument that this result is spurious given the issues with sample size and the disturbance values. In fact, any hypotheses or conclusions related to analyses of gender differences in this study must be treated tentatively. However, the gender difference in the relationship between working memory and computational fluency is similar to the gender difference found between these abilities in another, unpublished study (Tolar, 2005). Although no definitive conclusions can be made based on these results, they do provide justification for a new avenue of research.

One cognitive factor that was not included in the models of math achievement in this study was processing speed. Processing speed was not included because evidence suggests that it plays less of a role in math achievement than working memory among adolescents and adults (Floyd et al., 2003). However, a recently published study suggests that processing speed may have more of an effect on higher level math achievement (i.e.,

the SAT-M) than does working memory (Rohde & Thompson, 2007). Although the cognitive models in this study explained half of the variance in algebra achievement and more than half of the variance in SAT-M performance, there was still a substantial amount of unexplained variance. Processing speed may be one factor that adds more explanatory power to the cognitive models of math achievement examined in this study.

Conclusion

Math achievement is a complex construct that includes acquired knowledge and skills from different domains of math as well as fluid problem solving abilities. Math achievement is influenced by the coordination of domain specific and domain general cognitive resources. The ways in which these processes interact to influence achievement differ across definitions of math achievement, including definitions based on math domain (e.g., arithmetic vs. algebra) and definitions based on problem type (e.g., procedural vs. novel). Systematic comparisons of cognitive models of math achievement are an important step to developing educational programs that will improve student outcomes and to aligning educational programs with assessments of math achievement that have profound influences on educational and economic opportunities.

References

Baddeley, A., & Logie, R. H. (1999). Working memory: The multi-component model. In A. Miyake & P. Shah (Eds.), *Models of working memory: Mechanisms of active maintenance and executive control* (pp. 28-61). New York: Cambridge University Press.

Bell, A. (1996). Problem-solving approaches to algebra: Two aspects. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 167-185). Norwell, MA: Kluwer Academic Publishers.

Bennet, G. K., Seashore, H. G., & Wesman, A. G. (1989). *Differential aptitude tests for personnel and career assessment: Space relations*: The Psychological Corporation/Harcourt Brace & Company.

Bentler, P. M. (1990). Comparative fit indexes in structural models. *Psychological Bulletin, 107*, 238-246.

Blanton, M. L., & Kaput, J. J. (2002). *Instructional contexts that support students' transition from arithmetic to algebraic reasoning: Elements of tasks and culture.* Paper presented at the NCTM Annual Conference Research Pre-session, Las Vegas, NV.

Bull, R., & Johnston, R. S. (1997). Children's arithmetical difficulties: Contributions from processing speed, item identification, and short-term memory. *Journal of Experimental Child Psychology, 65*, 1-24.

Bull, R., Johnston, R. S., & Roy, J. A. (1999). Exploring the roles of the visual-spatial sketch pad and central executive in children's arithmetical skills: Views from cognition and developmental neuropsychology. *Developmental Neuropsychology, 15*(3), 421-442.

Bull, R., & Scerif, G. (2001). Executive functioning as a predictor of children's mathematics ability: Inhibition, switching, and working memory. *Developmental Neuropsychology, 19*(3), 273-293.

Campbell, J. I. D. (1990). Retrieval inhibition and interference in cognitive arithmetic. *Canadian Journal of Psychology, 44*, 445-464.

Campbell, J. I. D., & Arbuthnott, K. (1996). Inhibitory processes in sequential retrieval: Evidence from variable-lag repetition priming. *Brain and Cognition, 30*, 59-80.

Carry, L. R., Lewis, C., & Bernard, J. E. (1980). *Psychology of equation solving: An information processing study* (No. Tech. Rep. SED 78-22293). Austin: The University of Texas at Austin, Department of Curriculum and Instruction: The University of Texas at Austin, Department of Curriculum and Instruction.

Casey, M. B. (1996). Understanding individual differences in spatial ability within females: A nature/nurture interactionist framework. *Developmental Review, 16*, 241-260.

Casey, M. B., Nuttall, R. L., & Pezaris, E. (1997). Mediators of gender differences in mathematics college entrance test scores: A comparison of spatial skills with internalized beliefs and anxieties. *Developmental Psychology, 33*, 669-680.

Casey, M. B., Nuttall, R. L., Pezaris, E., & Benbow, C. P. (1995). The influence of spatial ability on gender differences in mathematics college entrance test scores across diverse samples. *Developmental Psychology, 31*, 697-705.

Cirino, P. T., Morris, M. K., & Morris, R. D. (2002). Neuropsychological concomitants of calculation skills in college students referred for learning difficulties. *Developmental Neuropsychology, 21*(2), 201-218.

Coley, R. J. (2001). *Differences in the gender gap: Comparisons across racial/ethnic groups in education and work*. Princeton, NJ: Educational Testing Service.

Colom, R., Contreras, M. J., Arend, I., Leal, O. G., & Santacreu, J. (2004). Sex differences in verbal reasoning are mediated by sex differences in spatial ability. *The Psychological Record, 54*, 365-372.

Conway, A. R. A., Cowen, N., Bunting, M. F., Therriault, D. J., & Minkoff, S. R. B. (2002). A latent variable analysis of working memory capacity, short-term memory capacity, processing speed, and general fluid intelligence. *Intelligence, 30*, 163-183.

Conway, A. R. A., Kane, M. J., Bunting, M. F., Hambrick, D. Z., Wilhelm, O., & Engle, R. W. (2005). Working memory span tasks: A methodological review and user's guide. *Psychonomic Bulletin & Review, 12*, 769-786.

Dark, V. J., & Benbow, C. P. (1990). Enhanced problem translation and short-term memory: Components of mathematical talent. *Journal of Educational Psychology, 82*, 420-429.

Dark, V. J., & Benbow, C. P. (1991). Differential enhancement of working memory with mathematical versus verbal precocity. *Journal of Educational Psychology, 83*(1), 48 - 60.

Demetriou, A., Christou, C., Spanoudis, G., & Platsidou, M. (2002). The development of mental processing: Efficiency, working memory, and thinking. *Monographs of the Society for Research in Child Development, 67*, 1-169.

DeStefano, D., & LeFevre, J.-A. (2004). The role of working memory in mental arithmetic. *European Journal of Cognitive Psychology, 16*, 353-386.

Ekstrom, R. B., French, J. W., Harman, H. H., & Derman, D. (1976). *Manual for kit of factor-referenced cognitive tests*. Princeton, NJ: Educational Testing Service.

Engle, R. W. (2002). Working memory capacity as executive attention. *Current Directions in Psychological Science, 11*(1), 19-23.

Engle, R. W., Tuholski, S. W., Laughlin, J. E., & Conway, A. (1999). Working memory, short-term memory, and general fluid intelligence: A latent-variable approach. *Journal of Experimental Psychology: General, 128*, 309-331.

English, L. D., & Halford, G. S. (1995). *Mathematical education: Models and processes*. Mahwah, NJ: Lawrence Erlbaum Associates.

English, L. D., & Sharry, P. V. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics, 30*, 135-157.

Espy, K. A., McDiarmid, M. M., Cwik, M. F., Stalets, M. M., Hamby, A., & Senn, T. E. (2004). The contribution of executive functions to emergent mathematical skills in preschool children. *Developmental Neuropsychology, 26*, 465-486.

Floyd, R. G., Evans, J. J., & McGrew, K. S. (2003). Relations between measures of cattell-horn-carroll cognitive abilities and mathematics achievement across the school-age years. *Psychology in the Schools, 40*, 155-171.

Foegen, A., Olson, J., & Perkmen, S. (2005). Reliability and criterion validity of four revised algebra measures in districts b and c (project aaims: Algebra assessment and instruction - meeting standards technical report # 7). from http://www.ci.hs.iastate.edu/aaims/Technical.html

Foster, A. G., Winters, L. J., Gell, J. M., Rath, J. N., & Gordon, B. W. (1995). *Merrill algebra 1: Applications and connections*. New York: Glencoe/McGraw-Hill.

Friedman, L. (1995). The space factor in mathematics: Gender differences. *Review of Educational Research, 65*, 22-50.

Gallagher, A. M. (1992). Sex differences in the performance of high-scoring examinees on the sat-m: (Educational Testing Services Rep. No. 92-2). Princeton, NJ: Educational Testing Service.

Gallagher, A. M., De Lisi, R., Holst, P. C., McGillicuddy-De Lisi, A. V., Morely, M., & Cahalan, C. (2000). Gender differences in advanced mathematical problem solving. *Journal of Experimental Child Psychology, 75*(165-190).

Gathercole, S. E., Pickering, S. J., Knight, C., & Stegmann, Z. (2004). Working memory skills and educational attainment: Evidence from national curriculum assessments at 7 and 14 years of age. *Applied Cognitive Psychology, 18*, 1-16.

Geary, D. C. (1993). Mathematical disabilities: Cognitive, neuropsychological, and genetic components. *Psychological Bulletin, 114*, 345-362.

Geary, D. C., Liu, F., Chen, G.-P., Saults, S. J., & Hoard, M. K. (1999). Contributions of computional fluency to cross-national differences in arithmetical reasoning abilities. *Journal of Educational Psychology*, 716-719.

Geary, D. C., Saults, S. J., Liu, F., & Hoard, M. K. (2000). Sex differences in spatial cognition, computational fluency, and arithmetical reasoning. *Journal of Experimental Child Psychology, 77*, 337-353.

Geary, D. C., & Widaman, K. (1992). Numerical cognition: On the convergence of componential and psychometric models. *Intelligence, 16*, 47-80.

Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of abstract thought: Transitions from arithmetic to algebra. *Educational Studies in Mathematics, 36*, 219-245.

Heid, M. K. (1996a). Reflections on mathematical modeling and the redefinition of algebraic thinking. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 221-223). Norwell, MA: Kluwer Academic Publishers.

Heid, M. K. (1996b). A technology-intensive functional approach to the emergence of algebraic thinking. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 239-255). Norwell, MA: Kluwer Academic Publishers.

Heid, M. K. (2002). How theories about the learning and knowing of mathematics can inform the use of cas in school mathematics: One perspective. *The International Journal of Computer Algebra in Mathematics Education, 9*, 95-112.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59-78.

Hyde, J. S., Fennema, E., & Lamon, S. J. (1990). Gender differences in mathematics performance: A meta-analysis. *Psychological Bulletin, 107*, 139-155.

Janvier, C. (1996). Modeling and the initiation into algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 225-236). Norwell, MA: Kluwer Academic Publishers.

Jarvis, H. L., & Gathercole, S. E. (2003). Verbal and non-verbal working memory and achievement on national curriculum tests at 11 and 14 years of age. *Educational and Child Psychology, 20*, 123-140.

Johanning, D. I. (2004). Supporting the development of algebraic thinking in middle school: A closer look at students' informal strategies. *Journal of Mathematical Behavior, 23*, 371-388.

Joreskog, K., & Sorbom, D. (2006). Lisrel 8.80 student edition. Lincolnwood, IL: Scientific Software International, Inc.

Kane, M. J., Hambrick, D. Z., Tuholski, S. W., Wilhelm, O., Payne, T. W., & Engle, R. W. (2004). The generality of working memory capacity: A latent-variable approach to verbal and visuospatial memory span and reasoning. *Journal of Experimental Psychology: General, 133*, 189-217.

Kaput, J. J. (1995a). Long term algebra reform: Democratizing access to big ideas. In J. J. Kaput & W. Blair (Eds.), *The algebra initiative collogium* (Vol. 1, pp. 33-49). Washington D. C.: Department of Education: Office of Research.

Kaput, J. J. (1995b). *A research base for algebra reform: Does one exist?* Paper presented at the the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Kaput, J. J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Erlbaum.

Kerkman, D. D., Wise, J. C., & Harwood, E. A. (2000). Impossible "mental rotation" problems: A measure of women's spatial abilities? *Learning and Individual Differences, 12*, 253-269.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 96-112). New York: Cambridge University Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-414). New York: MacMillan Publishing Company.

Kieran, C., Boileau, A., & Garancon, M. (1996). Introducing algebra by means of a technology-supported, functional approach. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 257-293). Norwell, MA: Kluwer Academic Publishers.

Kline, R. B. (2005). *Principles and practice of structural equation modeling*. New York: The Guilford Press.

Lee, K., Ng, S.-F., Ng, E.-L., & Lim, Z.-Y. (2004). Working memory and literacy as predictors of performance on algebraic word problems. *Journal of Experimental Child Psychology, 89*, 140-158.

Lehto, J. (1995). Working memory and school achievement in the ninth form. *Educational Psychology, 15*(3), 271-283.

Ma, X. (2005a). Early acceleration of students in mathematics: Does it promote growth and stability of growth in achievement across mathematical areas? *Contemporary Educational Psychology, 30*, 439-460.

Ma, X. (2005b). A longitudinal assessment of early acceleration of students in mathematics on growth in mathematics achievement. *Developmental Review, 25*, 104-132.

Mardia, K. V. (1970). Measures of multivariate skewness and kurtosis with applications. *Biometrika, 57*, 519-530.

Mardia, K. V., & Foster, K. (1983). Omnibus tests of multinormality based on skewness and kurtosis. *Communications in Statistics, 12*, 207-221.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics, 9*(2), 2-8.

Mason, J. (1996). Expressing generality and roots in algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Norwell, MA: Kluwer Academic Publishers.

McNeil, N. M., & Alibali, M. W. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development, 76*, 883-899.

Miyake, A., Friedman, N. P., Rettinger, D. A., Shah, P., & Hegarty, M. (2001). How are visuospatial working memory, executive functioning, and spatial abilities related? A latent-variable analysis. *Journal of Experimental Psychology: General, 130*, 621-640.

Nathan, M. J., & Koedinger, K. R. (2000). Teachers' and researchers' beliefs about the development of algebraic reasoning. *Journal for Research in Mathematics Education, 31*(2), 168-190.

Oberauer, K., Heinz-Martin, S., Wilhelm, O., & Werner, W. W. (2003). The multiple faces of working memory: Storage, processing, supervision, and coordination. *Intelligence, 31*, 167-193.

Paul, F. G. (2005). Grouping within algebra i: A structural sieve with powerful effects for low-income, minority, and immigrant students. *Educational Policy, 19*, 262-282.

Perlow, R., Moore, D. D. W., Kyle, R., & Killen, T. (1999). Convergent evidence among content-specific versions of working memory tests. *Educational and Psychological Measurement, 59*, 866-877.

Peters, M. (1995). *Revised vandenberg & kuse mental rotations tests: Forms mrt-a to mrt-d* (No. Technical Report, Department of Psychology, University of Guelph). Guelph (ON), Canada: Department of Psychology, University of Guelph.

Reuhkala, M. (2001). Mathematical skills in ninth-graders: Relationships with visuo-spatial abilities and working memory. *Educational Psychology, 21*(4), 387-399.

Robinson, N., Abbott, R. D., & Berninger, V. W. (1996). The structure of abilities in math-precocious young children: Gender similarities and differences. *Journal of Educational Psychology, 88*, 341-352.

Rohde, T. E., & Thompson, L. A. (2007). Predicting academinc achievement with cognitive ability. *Intelligence, 35*, 83-92.

Rojano, T. (1996). The role of problems and problem solving in the development of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 55-62). Norwell, MA: Kluwer Academic Publishers.

Royer, J. M., Tronsky, L. N., & Chan, Y. (1999). Math-fact retrieval as the cognitive mechanism underlying gender differences in math test performance. *Contemporary Educational Psychology, 24*, 181-266.

Scali, R. M., Brownlow, S., & Hicks, J. L. (2000). Gender differences in spatial task performance as a function of speed or accuracy orientation. *Sex Roles, 43*, 359-376.

Schoenfeld, A. H. (2004). Math wars. In B. C. Johnson & W. L. Boyd (Eds.), *Politics of education yearbook.*

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14*, 15-39.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning, 2*(3), 157-189.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - the case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Shah, P., & Miyake, A. (1996). The separability of working memory resources for spatial thinking and language processing: An individual differences approach. *Journal for Experimental Psychology: General, 125*, 4-27.

Shepard, S., & Metzler, D. (1978). Mental rotation: Effects of dimenionality of objects and types of tasks. *Journal of Experimental Psychology: Human Perception and Performance, 14*, 3-11.

Sjostrom, M. P. (2000). *Beliefs and practices of teachers regarding the high failure rate in algebra i.* Unpublished Dissertation, Georgia State University, Atlanta.

Sleeman, D. (1984). Solving linear algebraic equations. *Mathematics in School, 13*, 37-38.

Sleeman, D. (1986). Introductory algebra: A case study of student misconceptions. *Journal of Mathematical Behavior, 5*, 25-52.

Steen, L. A. (1999). Algebra for all in eighth grade: What's the rush? *Middle Matters, 8*, 6-7.

Steiger, J. H. (1990). Structural model evaluation and modifications: An interval estimation approach. *Multivariate Behavioral Research, 25*, 173-180.

Swafford, J. O., & Langrall, C. W. (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education, 31*(1), 89-112.

Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research, 76*, 249-274.

The College Board. (1989). *Guide to the use of the descriptive tests of mathematics skills*. Princeton, NJ: Educational Testing Service.

The College Board. (1995a). *Descriptive tests of mathematics skills in elementary algebra skills*. Princeton, NJ: Educational Testing Service.

The College Board. (1995b). *Descriptive tests of mathematics skills in intermediate algebra skills*. Princeton, NJ: Educational Testing Service.

The College Board. (2005). 2005 college-bound seniors: Total group profile report. Retrieved May 16, 2007, from http://www.collegeboard.com/prod_downloads/about/news_info/cbsenior/yr2005/2005-college-bound-seniors.pdf

Tolar, T. (2005). Gender differences in the relationship between working memory capacity, visual spatial abilities, and mathematics achievement among college students referred for learning difficulties: Georgia State University.

U.S. Department of Education. (1997). *Mathematics equals opportunity: White paper prepared for u.S. Secretary of education richard w. Riley* (No. ERIC Document Reproduction Service No. ED415119). Washingtion, D.C.: Department of Education.

Unsworth, N., Heitz, R. P., Schrock, J., & Engle, R. W. (2005). An automated version of the operation span task. *Behavior Research Methods, 37*, 498-505.

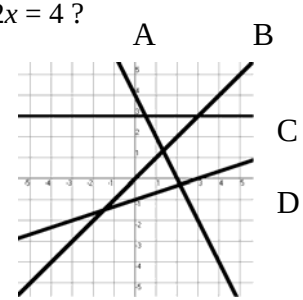
Vandenberg, S. G., & Kuse, A. R. (1978). Mental rotations, a group test of three-dimensional spatial visualization. *Perceptual and Motor Skills, 47*, 599-604.

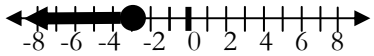
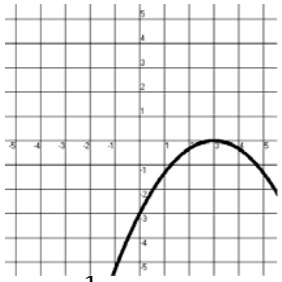
Voyer, D. (1996). The relation between mathematical achievement and gender differences in spatial abilities: A suppression effect. *Journal of Educational Psychology, 88*, 563-571.

Wechsler, D. (1997). *Wechsler adult intlligence scale - third edition*: The Psychological Corporation.

APPENDIXES

APPENDIX A: AAIMS Algebra Content Test

Evaluate $a^2 - b \div 2$ when $a = 4$ and $b = 6$ a) 1 b) 5 c) 10 d) 13	Simplify: $3(m + 2) + 2(m - 1)$ a) $5m + 4$ b) $5m + 1$ c) $6m + 8$ d) $6m - 8$	Simplify: $6(2b - 3) - 3(2 - b)$ a) $15b - 24$ b) $9b - 9$ c) $9b + 12$ d) $15b + 12$	Solve: $6c + 4 = -3c - 14$ a) $-\frac{10}{3}$ b) -2 c) 2 d) 6
Which line on the graph is $y + 2x = 4$?  a) Line A b) Line B c) Line C d) Line D	Find the slope of a line through (1, –1) (5, 2) a) $\frac{1}{5}$ b) $\frac{3}{4}$ c) -6 d) $-\frac{4}{3}$	Write the equation in slope-intercept form if $m = \frac{1}{2}$ and $b = 3$ a) $y = 2x + 3$ b) $y = 3x + \frac{1}{2}$ c) $x = \frac{1}{2}y - 3$ d) $y = \frac{1}{2}x + 3$	Write the equation of a line through (5, 3) (4, 9). Use point-slope form. a) $y + 1 = 2(x - 4)$ b) $y + 4 = -6(x - 1)$ c) $y - 3 = -6(x - 5)$ d) $y = -6x + 30$

This graph shows the solution for which equation?  a) $x > -3$ b) $2x \le -6$ c) $-3x > 9$ d) $3x \ge 9$	Which of the following is a logical first step to solve this linear system using substitution? $3x + 2y = 4$ $4x + y = 7$ a) $x = -2y + 4$ b) $y = -4x + 7$ c) $4 = 3x + 2y$ d) $4x = -y + 7$	Solve the linear system: $x - y = 4$ $x + 2y = 19$ a) $(-1, -5)$ b) $(5, 8)$ c) $(-2, 19)$ d) $(9, 5)$	Simplify, with no negative exponents: $\left(\frac{6x^2y^{-1}}{2xy}\right)^2$ a) $9x^2$ b) $3x^2y^3$ c) $\frac{3x}{y^2}$ d) $\frac{9x^2}{y^4}$
Simplify: $\sqrt{32}$ a) $4\sqrt{2}$ b) $8\sqrt{4}$ c) $\sqrt{16} \bullet \sqrt{2}$ d) $8\sqrt{2}$	Which function matches this graph?  a) $y = \frac{1}{3}x^2 - 7x - 2$ b) $y = x^2 + 2x + 3$ c) $y = -\frac{1}{3}x^2 + 2x - 3$ d) $y = -x^2 - 3$	Factor this trinomial: $2x^2 + 5x - 3$ a) $(x - 2)(x - 1)$ b) $(2x - 1)(x + 3)$ c) $(2x + 1)(x - 3)$ d) $(x - 1)(x + 3)$	Simplify: $\frac{12}{2x+4} + \frac{3x}{x+2}$ a) $\frac{3x+12}{3x+6}$ b) $\frac{x+4}{x+2}$ c) 3 d) $9x$

APPENDIX B: Algebra Equations Test

Solve the following equations. If you make an error, DO NOT ERASE. Draw one line through your error and continue.

$2x+3=5$	$2(x+1)=6$
$x+2(x+1)=4$	$7x+5=15-8x$

$7-2(x+5)=3x+7$	$9(x+40)=5(x+40)$
$\frac{5}{10}=\frac{x-10}{x+5}$	$\frac{1}{3}=\frac{1}{x}+\frac{1}{7}$

$xy + yz = 2y$ Solve for x	$\frac{x+3+x}{x^2} = 1$
$2x = x^2$	$A = p + prt$ Solve for p

APPENDIX C: Math Background Sheet

Age: _____

College Major: _____

Gender (please circle): Female Male

What math courses (or equivalent) did you take during <u>high school</u>? Please put year taken next to the course.

_____ Algebra I

_____ Algebra II

_____ Algebra III

_____ Advanced Algebra/Trigonometry

_____ Pre-calculus

_____ Calculus

Other _____

Please list the <u>college</u> math courses you have taken:

College Math Course	Semester/Year Taken

When did you take your most recent math class (semester/year)? _____

What was your most recent math class? _____

Do you have any diagnosed learning disabilities? _____ If yes, please describe: