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THEORETICAL AND NUMERICAL STUDY OF TIKHONOV'S REGULARIZATION AND MOROZOV'S DISCREPANCY PRINCIPLE

by

MARYGEORGE L. WHITNEY

Under the Direction of Dr. Alexandra Smirnova

ABSTRACT

A concept of a well-posed problem was initially introduced by J. Hadamard in 1923, who expressed the idea that every mathematical model should have a unique solution, stable with respect to noise in the input data. If at least one of those properties is violated, the problem is ill-posed (and unstable). There are numerous examples of ill-posed problems in computational mathematics and applications. Classical numerical algorithms, when used for an ill-posed model, turn out to be divergent. Hence one has to develop special regularization techniques, which take advantage of an a priori information (normally available), in order to solve an ill-posed problem in a stable fashion. In this thesis, theoretical and numerical investigation of Tikhonov's (variational) regularization is presented. The regularization parameter is computed by the discrepancy principle of Morozov, and a first-kind integral equation is used for numerical simulations.

INDEX WORDS: Tikhonov regularization, Morozov discrepancy principle, Ill-posed problems, Newton's method, Georgia State University

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MARYGEORGE L. WHITNEY

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2009

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2009

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This thesis is dedicated to Floyd Clifford Whitney III,

Dr. Alexandra Smirnova,

and Dr. Rachael Belinsky.

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Chapter 1

INTRODUCTION

The definition of a well-posed problem was originally introduced by J. Hadamard [6] in 1923 in order to figure out what types of boundary conditions are to be used for various types of differential equations. Hadamard's concept reflected the idea that any mathematical model of a physical problem must have the properties of uniqueness, existence, and stability of the solution. If at least one of these properties is not satisfied, the problem is ill-posed. Among classical ill-posed problems are stable numerical differentiation of noisy data, stable inversion of ill-conditioned matrices, parameter identification in partial differential equations, stable solution of first-kind integral equations, and many other examples.

Let X and Y be normed spaces and the problem

$$Kx = f,\ K : X \to Y \tag{1.1}$$

be ill-posed. Suppose the right-hand side f is given by its δ-approximation, f_δ, such that $\|f - f_\delta\| \leq \delta$. It would be natural to seek an approximate solution to (1.1) in the set $Q_\delta := \{x \in X : \|Kx - f_\delta\| \leq \delta\}$. However, in the ill-posed case an arbitrary element $x_\delta \in Q_\delta$ can not be used as an approximate solution to (1.1). Since x_δ does not depend continuously on f_δ, the fact that $\|Kx - f_\delta\| \leq \delta$ does not guarantee that x_δ is close to

the solution we are looking for.

Thus, if problem (1.1) is ill-posed, one has to use *a priori* information (normally available) in order to solve the problem in a stable manner, i.e., to construct a **regularized** solution to (1.1). Such *a priori* information as smoothness of the solution generates the technique, which is called **Tikhonov (variational)** regularization [15] [14]. This regularization allows one to obtain stable approximate solutions to ill-posed problems by means of a stabilizing functional. Initially proposed by A. Tikhonov, the variational methods have been further developed in [13], [5], [3], [8], [16].

One can also find approximate solutions to (1.1) by **iterations** (see [1], [17], [18], [7]), taking $x_n = F(f_\delta, x_{n-1}, ..., x_{n-k})$, where $k \leq n$. For these solutions to be stable under small changes in the initial data, the iteration number $n = n(\delta)$ yielding x_n must be compatible with the size of the error in the initial data.

Another important approach to the theory of ill-posed problems is based on the singular value decomposition (SVD) of a compact operator in a Hilbert space. Using SVD, one obtains an admissible regularization strategy by proper filtering of the singular system.

This thesis is organized as follows:

- Chapter 2. Examples of ill-posed problems and the regularization overview.
- Chapter 3. The Tikhonov regularization theory.
- Chapter 4. Numerical simulations.
- Chapter 5. Conclusions

Chapter 2

UNSTABLE MODELS

2.1 Examples of Ill-posed Problems

In this section, we give some fundamental results and definitions related to the theory of ill-posed problems. We also present and discuss some important examples of unstable mathematical models.

We start with the definition of a norm. The norm will be used to measure distance between two functions. In what follows, it will be the distance between exact and noisy data, and between exact and approximate solutions.

Definition 2.1.1. [2] Let X be a vector space over the field $\mathbb{R}$ ($\mathbb{C}$). X is <u>normed</u>, if there is a function $||\cdot||$: $X \to \mathbb{R}$ ($\mathbb{C}$), which we call a <u>norm</u>, satisfying all of the following:

1. $||f|| \geq 0$, and $||f|| = 0$ if and only if $f = 0$ (positive definiteness),

2. $||\alpha f|| = |\alpha| \, ||f||$ for any real (or complex) scalar α (homogeneity),

3. $||f + g|| \leq ||f|| + ||g||$ (the triangle inequality).

Definition 2.1.2. [9] Let X and Y be normed spaces, and $K : X \to Y$ (not necessarily onto). The problem of finding a solution $x \in X$ from the data $f \in Y$, $Kx = f$, is called <u>well-posed</u>, or properly posed if the following are all true:

1. Existence: For every $f \in Y$ there is (at least one) $x \in X$ such that $Kx = f$ (K is onto).

2. Uniqueness: For every $f \in Y$ there is at most one $x \in X$ with $Kx = f$ (K is one-to-one).

3. Stability: The solution x depends continuously on $f \in Y$, i.e., for every $\{x_n\} \subset X$ with $Kx_n \to Kx, (n \to \infty)$, it follows that $x_n \to x,\ (n \to \infty)$.

If at least one condition is violated then the problem is <u>ill-posed</u> or unstable.

The first example is one in partial differential equations involving an initial value problem for the Laplace equation. In fact, this is Cauchy's problem for the Laplace equation, a classic example that was done by Hadamard [6].

Example 2.1.3. The noisy data is denoted by f_δ and g_δ and the exact data is denoted by f_e and g_e. Consider the Laplace equation:

$$\Delta u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0, \quad x \in \mathbb{R}, \quad t \in [0, \infty)\,,$$

with the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad x \in \mathbb{R}.$$

Suppose

$$f_e(x) = g_e(x) = 0 \quad \text{and} \quad f_\delta(x) = 0, \quad g_\delta(x) = \delta \sin\left(\frac{x}{\delta}\right),\ \delta > 0$$

We now introduce the norm in the data space as $\|(f,g)\| := \sup_{x\in\mathbb{R}}\{|f| + |g|\}$, and the norm in the solution space as $\|u\| := \sup_{x\in\mathbb{R}} |u(x,t)|$. One has

$$\|(f_\delta - f_e, g_\delta - g_e)\| = \sup_{x\in\mathbb{R}}\{|f_\delta - f_e| + |g_\delta - g_e|\} = \sup_{x\in\mathbb{R}}\{|f_\delta| + |g_\delta|\}$$

$$= \sup_{x\in\mathbb{R}}\Big\{|0| + \Big|\delta\sin\Big(\frac{x}{\delta}\Big)\Big|\Big\} = \sup_{x\in\mathbb{R}}\Big\{\Big|\delta\sin\Big(\frac{x}{\delta}\Big)\Big|\Big\} = \delta.$$

For the noisy data, one can show that the corresponding solution

$$u_\delta(x,t) = \delta^2\sin\Big(\frac{x}{\delta}\Big)\sinh\Big(\frac{t}{\delta}\Big) \quad x\in\mathbb{R}, \quad t\geq 0.$$

Indeed, the partial derivatives are as follows:

$$(u_\delta)_x = \delta\cos\Big(\frac{x}{\delta}\Big)\sinh\Big(\frac{t}{\delta}\Big), \quad \text{and} \quad (u_\delta)_t = \delta\sin\Big(\frac{x}{\delta}\Big)\cosh\Big(\frac{t}{\delta}\Big).$$

The second partial derivatives are

$$(u_\delta)_{xx} = -\sin\Big(\frac{x}{\delta}\Big)\sinh\Big(\frac{t}{\delta}\Big), \quad \text{and} \quad (u_\delta)_{tt} = \sin\Big(\frac{x}{\delta}\Big)\sinh\Big(\frac{t}{\delta}\Big).$$

These derivatives will then give us $(u_\delta)_{xx} + (u_\delta)_{tt} = \Delta u_\delta(x,t) = 0$. For the exact data, $u_e(x,t) = 0,\ x\in\mathbb{R},\ t\geq 0$. Therefore

$$\begin{aligned}\|(u_\delta - u_e)\| &= \sup_{x\in\mathbb{R}}\Big|u_\delta(x,t) - u_e(x,t)\Big| = \sup_{x\in\mathbb{R}}\Big|\delta^2\sin\Big(\frac{x}{\delta}\Big)\sinh\Big(\frac{t}{\delta}\Big)\Big| = \delta^2\Big|\sinh\Big(\frac{t}{\delta}\Big)\Big| \\ &= \delta^2\frac{e^{\frac{t}{\delta}} - e^{\frac{-t}{\delta}}}{2}\end{aligned}$$

Let $\delta = \frac{1}{n}$, then for any $t > 0$, $\lim_{n\to\infty}\frac{e^{nt}-e^{-nt}}{2n^2} = (\frac{\infty}{\infty})$, since by repeated use of L'Hôpital's rule we end up with $\frac{t^2}{4}\lim_{n\to\infty} e^{nt} - e^{-nt} = \infty$. So, $\|(f_\delta - f_e, g_\delta - g_e)\| = \delta \to 0$, as $\delta\to 0$, while $\|u_\delta - u_e\|$ does not approach 0 as δ goes to 0. This means the problem is ill-posed.

In the following example we look at a differentiation problem to show how the slight change in the data can cause a considerable change in the solution. The noisy data is denoted by f_δ. The exact data is denoted by f_e.

Example 2.1.4. Let $f_e(x) = 0$ and $f_\delta(x) = \delta \sin(\frac{x}{\delta^2})$. Using the infinity norm we get

$$\|f_e - f_\delta\|_\infty = \sup_{x\in\mathbb{R}} \Big|0 - \delta \sin\Big(\frac{x}{\delta^2}\Big)\Big| = \sup_{x\in\mathbb{R}} |\delta| \Big|\sin\Big(\frac{x}{\delta^2}\Big)\Big| = \delta.$$

Now if we take the derivatives of $f(x)$ we get $f_e'(x) = 0$ and $f_\delta'(x) = \frac{1}{\delta}\cos(\frac{x}{\delta^2})$. Using the infinity norm in the solution space we then get $\|f_\delta' - f_e'\|_\infty = \sup_{x\in\mathbb{R}} |\frac{1}{\delta}\cos(\frac{x}{\delta^2}) - 0| = \frac{1}{\delta}$, and as δ goes to 0 the supremum goes to infinity. This means that the differentiation problem is unstable or ill-posed.

In the next example we use a matrix norm and a vector norm to analyze an ill-conditioned liner system $Ax = f$. Assume that A is a nonsingular matrix, f_δ is a perturbed (or noisy) right-hand side, and f_e is the exact right-hand side.

Example 2.1.5. Solving for x we get $x_e - x_\delta = A^{-1}(f_e - f_\delta)$, and solving for f_e we get $\|f_e\| = \|Ax_e\| \leq \|A\|\,\|x_e\| \ldots \Rightarrow \frac{1}{\|x_e\|} \leq \frac{\|A\|}{\|f_e\|}$. We will use these facts while solving for the relative error as follows:

$$\begin{aligned}\frac{\|x_e - x_\delta\|}{\|x_e\|} &\leq \frac{\|A^{-1}\|\,\|f_e - f_\delta\|}{\|x_e\|} = \|A^{-1}\| \frac{\|f_e - f_\delta\|}{\|f_e\|}\|A\| \\ &\leq \underbrace{\|A\|\,\|A^{-1}\|}_{\text{cond}(A)} \frac{\|f_e - f_\delta\|}{\|f_e\|}.\end{aligned}$$

One can see from above that a large condition number can result in a considerable change in the solution even if the relative change in the right-hand side is small (unstable problem). In fact, cond(A) measures how close A is to a singular matrix. The following theorem helps to explain.

Theorem 2.1.6. *[4] Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Then for any singular matrix $B \in \mathbb{R}^{n\times n}$, $\frac{1}{cond(A)} \leq \frac{\|A-B\|}{\|A\|}$.*

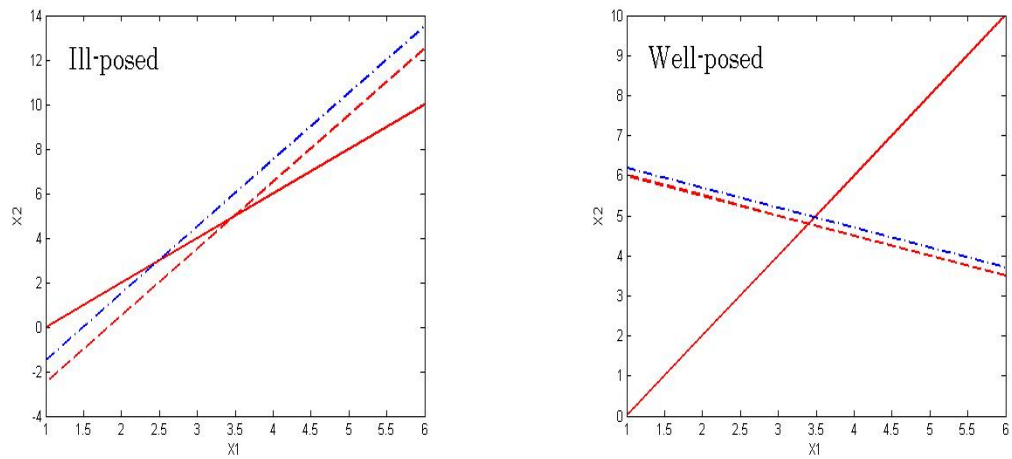

Figure 2.1. Ill-posed and Well-posed linear systems

So, as A gets close to singular, cond(A) approaches infinity. In the two dimensional case represented by a two-by-two system,

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$$\Leftrightarrow \vec{a_1}x_1 + \vec{a_2}x_2 = \vec{b} \Leftrightarrow \vec{a_1}x_1 = \vec{b} - \vec{a_2}x_2$$

i.e., the solution is equivalent to finding the intersection of the one dimensional subspace $\{x\vec{a_1} : x \in \mathbb{R}\}$ with the affine subspace $\{\vec{b} - \vec{a_2}x : x \in \mathbb{R}\}$. Such intersections are shown in Figure 2.1. If the subspaces are nearly parallel to the solution then the condition number is large since the matrix is almost singular, and a slight perturbation in the system may result in a considerable change of the solution. On the other hand, if the subspaces are almost perpendicular to the solution, a small perturbation or slight shift means you will still get a good solution.

Next we go to linear equations with compact operators $Kx = f$ using the following

definition:

Definition 2.1.7. K is a compact operator if it is a linear operator from a Banach space X to a Banach space Y such that the image under K for any bounded subset of X is a relatively compact set (its closure is compact).

Theorem 2.1.8. *[9] Linear equations $Kx = f$ with compact operators $K : X \to Y$, where X and Y are normed spaces and $\dim X = \infty$ are always ill-posed.*

Proof:

To show that $Kx = f$ is ill-posed, we will prove that K^{-1} is unbounded. Assume the converse: K^{-1} exists and is bounded, then $K^{-1}K = I$ should be compact since a superposition of a bounded and compact operators is a compact operator. Contradiction, since a unit ball in infinite dimensional space is not a compact set.$\square$

The following is an example of an ill-posed equation with a compact operator:

$$Kx := \int_a^b k(t,s)x(s)ds = f(t), \quad K : X \to Y, \quad t \in (c,d).$$

This comes from the following theorem:

Theorem 2.1.9. *[9]*

(a) Let $k(t,s) \in L^2\Big((c,d) \times (a,b)\Big)$. The operator $K : L^2(a,b) \to L^2(c,d)$, defined by

$$Kx := \int_a^b k(t,s)x(s)ds, \quad t \in (c,d), \quad x \in L^2(a,b),$$

is compact from $L^2(a,b)$ into $L^2(c,d)$.

(b) Let $k(t,s)$ be continuous on $[c,d] \times [a,b]$. Then K defined by the integral above is also compact as an operator from $C[a,b]$ into $C[c,d]$.

2.2 Numerical Aspects

The following example was initially done by Kirsch, [9], but is done here in a different way. While the conclusions are the same, the outcomes and methods used are different. In both cases, the results lead to the conclusion that the problem is ill-posed, but the numerical outcomes and the way in which they were obtained are different. The Matlab code used and the results can be seen in Appendix A and B respectively.

Example 2.2.1. [9] Consider the integral equation

$$Kx := \int_0^1 e^{ts} x(s) ds = f(t), \quad K : C[0,1] \to C[0,1],$$

where $0 \leq t \leq 1$. Let $f(t) = \frac{e^{t+1}-1}{t+1}$. The unique solution is $x(t) = e^t$. We are to approximate the integral using the trapezoidal rule as follows:

$$\int_0^1 e^{ts} x(s) ds \approx h\left[\frac{1}{2}x(0) + \frac{1}{2}e^t x(1) + \sum_{j=1}^{n-1} e^{jht} x(jh)\right] \approx f(t),$$

where $h = \frac{1}{n}$ and $t = ih$. The vector $[x_0 \ x_1 \ \ldots x_n]^T$ represents the numerical solution given the right-hand side $[f_0 \ f_1 \ \ldots f_n]^T$. For the experiment we take $n = 4, 8, 16, 32$ as the number of partitions and $t = 0, 0.25, 0.5, 1$ to compare the solutions corresponding to different values of n. Using a table for n vs. t we are to compute $x(t_i) - x_i$ in Matlab. See Appendix A for Matlab code and Appendix B for the results for $n = 16$, and $n = 32$. The Figure 2.2 illustrates the numerical results for $n = 4$ and $n = 8$ as compared to the exact solution. Please note that in the Matlab code, the statement $x = K\backslash f$ means that x is determined by the division of K into f, which is roughly the same as $inv(K) * f$ except that it is calculated by Gaussian elimination and forward/backward substitution, with no explicit computation of the inverse.

Table 2.1. Comparative Results

	$x(t_i) - x_i$				
t	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$x(t_i) = exp(t_i)$
0	0.56	0.53	-3.20	-31.14	1.00
0.25	1.95	-0.74	45.30	13.40	1.28
0.50	0.69	-3.11	24.17	-12.32	1.65
0.75	3.13	-0.95	66.94	33.04	2.12
1.0	1.63	1.49	-0.72	-9.83	2.72

The table given in Kirsch [9] was done by calculating the $inv(K)$ and then multiplying by f instead of using the Gaussian elimination. This is known to be numerically unstable. Since Matlab does not calculate the $inv(K)$ our solutions are more accurate than those given by Kirsch. They agree with those computed by Kirsch for $n = 4$ only. Still, as n gets larger the approximation gets worse, to the point that matrix K becomes nearly singular at $n = 16$, causing a very bad approximation that gets even worse at $n = 32$. See Table 2.1 and Appendix B for results of the exercise.

Since K^{-1} is unbounded for the original infinite-dimensional problem, the matrix becomes closer to singular as the number of partitions increases. The finer the discretization, the worse the approximation, i.e., the problem is ill-posed.

2.3 A General Regularization Strategy

To present the general concept of a regularization strategy, we begin with the definition:

Definition 2.3.1. [9] A <u>regularizing strategy</u> is a family of linear and bounded operators $R_\alpha : Y \to X,\ \alpha > 0$ such that $\lim_{\alpha \to 0} \|R_\alpha K x - x\| = 0$ for all $x \in X$, i.e., the operators $R_\alpha K$ converge pointwise to the identity.

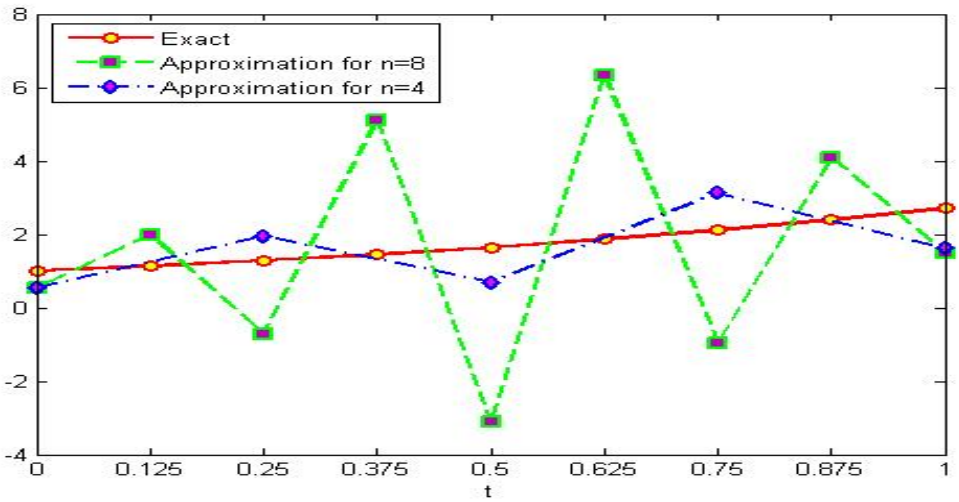

Figure 2.2. Exact and Approximate Solutions

Lemma 2.3.2. *[9] Let R_α be a regularizing strategy for a compact operator $K : X \to Y$, and $\dim X = \infty$, then the family R_α is not uniformly bounded, i.e., there exists $\{\alpha_j\}$, $\lim_{j\to\infty} \alpha_j = 0$, with $\|R_{\alpha_j}\| \to \infty$ for $j \to \infty$.*

Proof:

Assume the converse: there exists c such that for all $\alpha, \|R_\alpha\| \leq c$. Then for any $f \in K(X) \subset Y$, one has

$$\|K^{-1}f\| = \|R_\alpha f - R_\alpha f + K^{-1}f\| \leq \|K^{-1}f - R_\alpha f\| + \|R_\alpha f\| \leq \|x - R_\alpha Kx\|$$

$$+\|R_\alpha\|\|f\| \leq \|x - R_\alpha Kx\| + c\|f\|.$$

Pass to the limit as $\alpha \to 0$ and conclude that $\|K^{-1}f\| \leq c\|f\|$ since by definition $\lim_{\alpha\to 0} \|R_\alpha Kx - x\| = 0$. This implies $\|K^{-1}\| \leq c$, that is K^{-1} is bounded which is a contradiction. $\square$

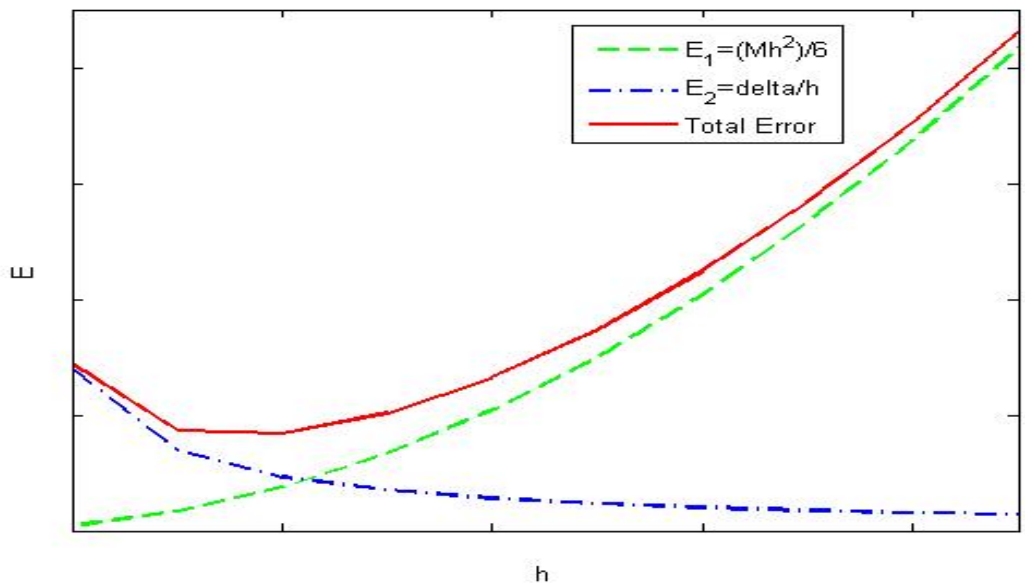

Figure 2.3. Error Estimation for the Central Difference Formula

Let $x_{\alpha,\delta} := R_\alpha f_\delta$. Then $E(\alpha) := \|x_{\alpha,\delta} - x_e\|$ estimates how good the approximation is for f_δ. Suppose $\|f_\delta - f_e\| \leq \delta$. Then

$$E(\alpha) = \|x_{\alpha,\delta} - x_e\| = \|R_\alpha f_\delta - R_\alpha f_e + R_\alpha f_e - x_e\| \leq \|R_\alpha f_\delta - R_\alpha f_e\| + \|R_\alpha f_e - x_e\|$$

$$\leq \|R_\alpha(f_\delta - f_e)\| + \|R_\alpha f_e - x_e\| \leq \|R_\alpha\|\|f_\delta - f_e\| + \|R_\alpha f_e - x_e\|$$

$$\leq \|R_\alpha\|\delta + \|R_\alpha K x_e - x_e\|. \tag{2.3.2.1}$$

Note as α approaches 0, $\|R_\alpha\|\delta$ approaches infinity and $\|R_\alpha K x_e - x_e\|$ approaches 0. This is the fundamental estimate for a regularization strategy.

We illustrate this estimate by constructing a regularization strategy for the numerical differentiation problem using the central difference formula and the step size h as a regularization parameter.

Example 2.3.3. Let the derivative be approximated at $t_0 \in (a, b)$, and h be the step size. Suppose $\|f_\delta - f_e\| = \sup_{t\in[t_0-h,t_0+h]} \|f_\delta(t) - f_e(t)\| = \delta$ and $R_h f_\delta(t_0) = \frac{f_\delta(t_0+h)-f_\delta(t_0-h)}{2h}$. By using the central difference formula one gets

$$E(h) := \left|f_e'(t_0) - R_h f_\delta(t_0)\right| = \left|f_e'(t_0) - \frac{f_\delta(t_0+h) - f_\delta(t_0-h)}{2h}\right|$$

$$= \left|f_e'(t_0) - \frac{f_\delta(t_0+h) - f_\delta(t_0-h)}{2h} + \frac{f_e(t_0+h) - f_e(t_0+h) - f_e(t_0-h) + f_e(t_0-h)}{2h}\right|$$

$$\leq \left|f_e'(t_0) - \frac{f_e(t_0+h) - f_e(t_0-h)}{2h}\right| + \frac{|f_\delta(t_0+h) - f_e(t_0+h)|}{2h} + \frac{|f_\delta(t_0-h) - f_e(t_0-h)|}{2h}.$$

Let us estimate the first term that measures the accuracy of the central difference formula. One has

$$f_e(t_0+h) = f_e(t_0) + f_e'(t_0)h + \frac{f_e''(t_0)h^2}{2} + \frac{f_e'''(\xi)h^3}{3!}, \quad \xi \in [t_0, t_0+h],$$

$$\text{and} \quad f_e(t_0-h) = f_e(t_0) - f_e'(t_0)h + \frac{f_e''(t_0)h^2}{2} - \frac{f_e'''(\mu)h^3}{3!}, \quad \mu \in [t_0-h, t_0].$$

By subtracting the two and using the mean value theorem one derives

$$f_e(t_0+h) - f_e(t_0-h) = 2f_e'(t_0)h + \frac{f_e'''(\eta)h^3}{3}, \eta \in [t_0-h, t_0+h].$$

Solving for the $\frac{f_e'''(\eta)h^2}{6}$ we get the following:

$$\frac{f_e'''(\eta)h^2}{6} = \frac{f_e(t_0+h) - f_e(t_0-h)}{2h} - f_e'(t_0).$$

Thus,

$$\left|f_e'(t_0) - \frac{f_e(t_0+h) - f_e(t_0-h)}{2h}\right| \leq \frac{M_3 h^2}{6},$$

if $\sup_{t\in[t_0-h,t_0+h]} |f'''(t)| = M_3$. The second and third terms in the estimate for $E(h)$ are

bounded by $\frac{\delta}{2h}$. Hence

$$E(h) \leq \frac{M_3 h^2}{6} + \frac{\delta}{h},\ \|R_h f_e - f_e'\| \leq \frac{M_3 h^2}{6} \to 0 \text{ and } \|R_\alpha(f_\delta - f_e)\| \leq \frac{\delta}{h} \to \infty.$$

Now if we take the derivative of $E(h) = \frac{M_3 h^2}{6} + \frac{\delta}{h}$ with respect to h and equate it to zero, we will get the optimal value of the regularization parameter

$$E'(h) = \frac{M_3 h}{3} - \frac{\delta}{h^2} = \frac{M_3 h^3 - 3\delta}{3h^2} = 0.$$

Solving for h we then have $M_3 h^3 - 3\delta = 0$, finally getting $h_{opt} = \left(\frac{3\delta}{M_3}\right)^{1/3}$ as can be seen in Figure 2.3. Note that h_{opt}, which plays the role of α in this example, is the point where the minimum of $E(h)$ is attained. Therefore $E(h_{opt}) = \left(\frac{9}{8} M_3 \delta^2\right)^{1/3}$.

Chapter 3

THE REGULARIZATION THEORY

3.1 The Tikhonov (Variational) Regularization

We begin with some important definitions:

Definition 3.1.1. [9] Let X be a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A <u>scalar product</u> or <u>inner product</u> is a mapping

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$$

with the following properties:

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$,

(ii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in X$ and $\alpha \in \mathbb{K}$,

(iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$,

(iv) $\langle x, x \rangle \in \mathbb{R}$ and $\langle x, x \rangle \geq 0$, for all $x \in X$,

(v) $\langle x, x \rangle > 0$ if $x \neq 0$.

The following properties are easily derived from the definition:

(vi) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$ for all $x, y, z \in X$,

(vii) $\langle x, \alpha y\rangle = \bar{\alpha}\,\langle x, y\rangle$ for all $x, y \in X$ and $\alpha \in \mathbb{K}$.

Definition 3.1.2. [9] A vector space X over $\mathbb{K}$ with inner product $\langle\cdot,\cdot\rangle$ is called a pre-Hilbert space over $\mathbb{K}$.

Definition 3.1.3. [9] A normed space X over $\mathbb{K}$ is called *complete* or a Banach space if every Cauchy sequence converges in X. A complete pre-Hilbert space is called a Hilbert space.

Definition 3.1.4. [9] Let : $K : X \to Y$ be a linear bounded operator between Hilbert spaces. Then there exists a unique linear bounded operator $K^* : Y \to X$ with the property $\langle Kx, y\rangle = \langle x, K^*y\rangle$ for all $x \in X$, $y \in Y$. The operator $K^* : Y \to X$ is called the adjoint operator to K. For $X = Y$, the operator K is called self-adjoint if $K^* = K$.

A common approach to overdetermined linear systems of the form $Kx = f$ is to solve them in the sense of least squares, i.e., to minimize $\|Kx - f\|$ with respect to $x \in X$. If X is infinite-dimensional and K is a compact operator, the above minimization problem is also ill-posed by the following lemma.

Lemma 3.1.5. *[9] Let X, Y be Hilbert spaces such that $K : X \to Y$, is linear and bounded. Then there exists $\hat{x} \in X$ such that $\hat{x} = \arg\min_{x\in X} \|Kx - f\|$, where* arg min *is the element that minimizes $\|Kx - f\|$, if and only if $\hat{x}$ solves the normal equation $K^*K\hat{x} = K^*f$. $K^* : Y \to X$.*

Proof:

$$
\begin{aligned}
\|Kx-f\|^2-\|K\hat{x}-f\|^2 &= \langle Kx-f, Kx-f\rangle - \|K\hat{x}-f\|^2 \\
&= \langle K(x-\hat{x})+K\hat{x}-f, K(x-\hat{x})+K\hat{x}-f\rangle - \|K\hat{x}-f\|^2 \\
&= \|K(x-\hat{x})\|^2 + \langle K(x-\hat{x}), K\hat{x}-f\rangle + \\
&\quad + \langle K\hat{x}-f, K(x-\hat{x})\rangle + \|K\hat{x}-f\|^2 - \|K\hat{x}-f\|^2 \\
&= \|K(x-\hat{x})\|^2 + \langle K(x-\hat{x}), K\hat{x}-f\rangle + \langle K\hat{x}-f, K(x-\hat{x})\rangle \\
&= \|K(x-\hat{x})\|^2 + \overline{\langle K\hat{x}-f, K(x-\hat{x})\rangle} + \langle K\hat{x}-f, K(x-\hat{x})\rangle \\
&= \|K(x-\hat{x})\|^2 + 2\Re e\,\langle K\hat{x}-f, K(x-\hat{x})\rangle \\
&= \|K(x-\hat{x})\|^2 + 2\Re e\,\langle K^*(K\hat{x}-f), x-\hat{x}\rangle\,.
\end{aligned}
$$

1. If $K^*K\hat{x} = K^*f$, then $2\Re e\,\langle K^*(K\hat{x}-f), x-\hat{x}\rangle = 0$ and $\|Kx-f\|^2 - \|K\hat{x}-f\|^2 = \|K(x-\hat{x})\|^2 \geq 0$, that is $\hat{x}$ minimizes $\|Kx-f\|$.

2. Now let $\hat{x}$ be a minimizer of $\|Kx-f\|$. Substitute $x = \hat{x}+tz$, $z \in X$, $t > 0$, then

$$
0 \leq \|K(x-\hat{x})\|^2 + 2\Re e\,\langle K\hat{x}-f, K(x-\hat{x})\rangle
$$

$$
= \|K(tz)\|^2 + 2\Re e\,\langle K\hat{x}-f, K(tz)\rangle
$$

$$
= t^2\|Kz\|^2 + 2t\Re e\,\langle K\hat{x}-f, Kz\rangle\,.
$$

Divide by t and pass to the limit as t approaches 0 to get:

$$
0 \leq t\|Kz\|^2 + 2\Re e\,\langle K\hat{x}-f, Kz\rangle\,,
$$

$$
0 \leq \lim_{t\to 0}\{t\|Kz\|^2 + 2\Re e\,\langle K\hat{x}-f, Kz\rangle\} = 2\Re e\,\langle K\hat{x}-f, Kz\rangle\,.
$$

Hence for all $z \in X$

$$0 \leq 2\Re e \langle K\hat{x} - f, Kz \rangle = 2\Re e \langle K^*(K\hat{x} - f), z \rangle .$$

Take, $z = -K^*(K\hat{x} - f)$. Then

$$0 \leq -\Re e \langle K^*(K\hat{x} - f), K^*(K\hat{x} - f) \rangle = -\|K^*(K\hat{x} - f)\|^2,$$

which means $\hat{x}$ solves the normal equation $K^*(K\hat{x} - f) = 0 \quad \square$

As the consequence of this lemma we should either modify the functional $\|Kx - f\|$ or rewrite the equation $K^*K\hat{x} = K^*f$ in such a way that the operator is no longer compact. Both ideas lead to the following (regularized) minimization problem:

$$J_\alpha(x) := \|Kx - f\|^2 + \alpha\|x\|^2 \text{ for } x \in X.$$

J_α is called the **Tikhonov functional**.

Theorem 3.1.6. *[9] Let $K : X \to Y$ be a linear bounded operator between Hilbert spaces and $\alpha > 0$. Then J_α has a unique minimum $x_\alpha \in X$. This minimum is the unique solution of the normal equation $\alpha x_\alpha + K^*Kx_\alpha = K^*f$.*

From the above identity it follows that $x_\alpha = (\alpha I + K^*K)^{-1}K^*f := R_\alpha f,\ R_\alpha : Y \to X$, where $R_\alpha := (\alpha I + K^*K)^{-1}K^*$. Suppose now that f is given by its $\delta-$approximation, i.e., $\|f_\delta - f_e\| \leq \delta$ and one solves the equation $Kx = f_\delta$. Then one gets

$$x_{\alpha,\delta} = (\alpha I + K^*K)^{-1}K^*f_\delta := R_\alpha f_\delta.$$

By our fundamental estimate $\|x_{\alpha,\delta} - x_e\| \leq \delta\|R_\alpha\| + \|R_\alpha Kx_e - x_e\|$ (see 2.3.2.1), where x_e solves the exact equation $Kx = f_e$. For $\|R_\alpha\|$, one obtains, $\|R_\alpha\| = \|(\alpha I + K^*K)^{-1}K^*\|$.

By polar decomposition $K = U(K^*K)^{1/2}$, where U is a partial isometry:

$$\|Uq\| = \|q\| \text{ for all } q \in N(U)^{\perp} \text{ and } N(U) = \{q : Uq = 0\}.$$

Therefore,

$$\|R_\alpha\| = \|(\alpha I + K^*K)^{-1}(K^*K)^{1/2}U\| \leq \|(\alpha I + K^*K)^{-1}(K^*K)^{1/2}\| := \|\varphi(A)\|$$

with $A := K^*K$. Since A is self-adjoint, by spectral theorem one derives

$$\|\varphi(A)\| = \sup_{\lambda \in \sigma(A)} |f(\lambda)| = \sup_{\lambda \in \sigma(A)} \frac{\sqrt{\lambda}}{\lambda + \alpha} = \frac{1}{2\sqrt{\alpha}}.$$

Here $\sigma(A) = [0, \|K\|^2]$. Thus, $\delta\|R_\alpha\| \leq \frac{\delta}{2\sqrt{\alpha}}$.

In order to analyze $\|R_\alpha K x_e - x_e\|$, we make an additional assumption: $x_e = K^*z \in K^*(Y),\ z \in Y$. Then

$$\begin{aligned}
\|R_\alpha K x_e - x_e\| &= \|[K^*K + \alpha I]^{-1}K^*Kx_e - x_e\| \\
&= \|[K^*K + \alpha I]^{-1}K^*Kx_e - [K^*K + \alpha I]^{-1}[K^*K + \alpha I]x_e\| \\
&= \|[K^*K + \alpha I]^{-1}(K^*Kx_e - K^*Kx_e - \alpha I x_e)\| \\
&= \|[K^*K + \alpha I]^{-1}(-\alpha I x_e)\| = \|[K^*K + \alpha I]^{-1}(-\alpha x_e)\| \\
&= \alpha\|[K^*K + \alpha I]^{-1}K^*z\| \leq \alpha\|[K^*K + \alpha I]^{-1}K^*\|\,\|z\| \\
&\leq \frac{\alpha\|z\|}{2\sqrt{\alpha}} = \frac{\|z\|\sqrt{\alpha}}{2}.
\end{aligned}$$

This inequality proves that the operators $R_\alpha : Y \to X,\ R_\alpha = (\alpha I + K^*K)^{-1}K^*$ form a regularization strategy with $\lim_{\alpha\to 0}\|R_\alpha K x_e - x_e\| \leq \lim_{\alpha\to 0}\frac{\|z\|\sqrt{\alpha}}{2} = 0$. It is called the **Tikhonov regularization** method. Combining the estimates for $\delta\|R_\alpha\|$ and $\|R_\alpha K x_e -$

$x_e\|$, one concludes

$$\|x_{\alpha,\delta} - x_e\| \leq \frac{\delta}{2\sqrt{\alpha}} + \frac{\|z\|\sqrt{\alpha}}{2} := E(\alpha).$$

Formally, one can now minimize $E(\alpha)$ in order to find the optimal value of the regularization parameter. However, $\|z\|$ is unknown. In the next section we discuss a posteriori choice of α by the so called discrepancy principle, which does not use $\|z\|$.

3.2 The Discrepancy Principle of Morozov

In this section, we consider the determination of $\alpha(\delta)$ from the discrepancy principle (DP) [10], [11], [12] of Morozov. The DP suggests computing $\alpha = \alpha(\delta) > 0$ in such a way that the corresponding Tikhonov solution:

$$\alpha x_{\alpha,\delta} + K^* K x = K^* f_\delta$$

that is, the minimum of the functional

$$J_{\alpha,\delta}(x) := \|Kx - f_\delta\|^2 + \alpha\|x\|^2,$$

satisfies the equation

$$\|Kx_{\alpha,\delta} - f_\delta\| = \delta.$$

This choice of α guarantees that, on one hand, the discrepancy is equal to δ and, on the other hand, α is not too small. Uniqueness and existence of the solution to $\|Kx_{\alpha,\delta} - f_\delta\| = \delta$ is justified by the following theorem.

Theorem 3.2.1. *Let X and Y be Hilbert spaces and $K : X \to Y$ be linear, compact, and one-to-one with a dense range in Y. Also, let $x_e = K^* z \in K^*(Y)$ be the exact solution to $Kx_e = f_e$ and $\|f_e - f_\delta\| \leq \delta < \|f_\delta\|$. If $\|Kx_{\alpha,\delta} - f_\delta\| = \delta$, then $\|x_{\alpha,\delta} - x_e\| \leq 2\sqrt{\delta\|z\|}$.*

Proof: Since $x_{\alpha,\delta}$ minimizes the Tikhonov functional $J_{\alpha,\delta} = \|Kx_e - f_\delta\|^2 + \alpha\|x_e\|^2$, we can conclude that

$$\alpha\|x_{\alpha,\delta}\|^2 + \delta^2 = J_{\alpha,\delta}(x_{\alpha,\delta}) \leq J_{\alpha,\delta}(x_e) = \alpha\|x_e\|^2 + \|f_e - f_\delta\|^2 \leq \alpha\|x_e\|^2 + \delta^2,$$

and hence $\|x_{\alpha,\delta}\| \leq \|x_e\|$ for all $\delta > 0$. This gives us the following important estimate:

$$\begin{aligned} \|x_{\alpha,\delta} - x_e\|^2 &= \|x_{\alpha,\delta}\|^2 - 2\Re e\,\langle x_{\alpha,\delta}, x_e\rangle + \|x_e\|^2 \\ &\leq 2[\|x_e\|^2 - \Re e\,\langle x_{\alpha,\delta}, x_e\rangle] = 2\Re e\,\langle x_e - x_{\alpha,\delta}, x_e\rangle\,. \end{aligned}$$

Let $x_e = K^*z, z \in Y$. Then

$$\begin{aligned} \|x_{\alpha,\delta} - x_e\|^2 &\leq 2\Re e\,\langle x_e - x_{\alpha,\delta}, K^*z\rangle = 2\Re e\,\langle f_e - Kx_{\alpha,\delta}, z\rangle \\ &\leq 2\Re e\,\langle f_e - f_\delta, z\rangle + 2\Re e\,\langle f_\delta - Kx_{\alpha,\delta}, z\rangle \\ &\leq 2\delta\|z\| + 2\delta\|z\| = 4\delta\|z\|. \end{aligned}$$

Therefore, $\|x_{\alpha,\delta} - x_e\| \leq 2\sqrt{\delta\|z\|}$, as was to be shown. $\square$

The condition $\|f_\delta\| > \delta$ is reasonable, since otherwise the right-hand side is less than δ, and one can take $x_\delta = 0$. This also shows that the determination of α does not have to satisfy the equation $\|Kx_{\alpha,\delta} - f_\delta\| = \delta$ exactly. We can use the following bounds that will then guarantee the same level of accuracy.

$$c_1\delta \leq \|Kx_{\alpha,\delta} - f_\delta\| \leq c_2\delta.$$

How to solve the equation $\|Kx_{\alpha,\delta} - f_\delta\| = \delta$? We introduce

$$\varphi(\alpha) := \|Kx_{\alpha,\delta} - f_\delta\|^2 - \delta^2.$$

Then the equation $\varphi(\alpha) = 0$ is equivalent to $\|Kx_{\alpha,\delta} - f_\delta\| = \delta$, and it can be solved numerically for example, by Newton's method:

$$\alpha_{j+1} = \alpha_j - \frac{\varphi(\alpha_j)}{\varphi'(\alpha_j)},\ j = 0, 1, 2, ...$$

The derivative $\varphi'(\alpha)$ can be calculated as follows:

$$\begin{aligned}
\varphi'(\alpha) &= \left[\langle Kx_{\alpha,\delta} - f_\delta, Kx_{\alpha,\delta} - f_\delta\rangle - \delta^2\right]'_\alpha \\
&= \left\langle K\frac{dx_{\alpha,\delta}}{d\alpha}, Kx_{\alpha,\delta} - f_\delta\right\rangle + \left\langle Kx_{\alpha,\delta} - f_\delta, K\frac{dx_{\alpha,\delta}}{d\alpha}\right\rangle \\
&= 2\Re e\left\langle K\frac{dx_{\alpha,\delta}}{d\alpha}, Kx_{\alpha,\delta} - f_\delta\right\rangle.
\end{aligned}$$

We now compute $\frac{dx_{\alpha,\delta}}{d\alpha}$ by differentiating the identity

$$\alpha x_{\alpha,\delta} + K^*Kx_{\alpha,\delta} = K^*f_\delta$$

implicitly. Since the right-hand side does not depend on α, one has

$$x_{\alpha,\delta} + \alpha\frac{dx_{\alpha,\delta}}{d\alpha} + K^*K\frac{dx_{\alpha,\delta}}{d\alpha} = 0.$$

Solving for $\frac{dx_{\alpha,\delta}}{d\alpha}$, we get

$$\frac{dx_{\alpha,\delta}}{d\alpha} = -[\alpha I + K^*K]^{-1}x_{\alpha,\delta} = -[\alpha I + K^*K]^{-1}[\alpha I + K^*K]^{-1}K^*f_\delta.$$

Substituting this into $\varphi'(\alpha)$, one derives

$$\begin{aligned}\varphi'(\alpha) &= 2\Re e\left\langle -K[\alpha I + K^*K]^{-1}x_{\alpha,\delta}, Kx_{\alpha,\delta} - f_\delta\right\rangle \\ &= 2\Re e\left\langle -K[\alpha I + K^*K]^{-1}[\alpha I + K^*K]^{-1}K^*f_\delta, K[\alpha I + K^*K]^{-1}K^*f_\delta - f_\delta\right\rangle.\end{aligned}$$

Chapter 4

NUMERICAL SIMULATIONS

4.1 Statement of the Problem

In this chapter, we will show that as opposed to the 'naive' discretization used in section 2.1, the Tikhonov regularization combined with Morozov's discrepancy principle, results in a stable numerical solution to an ill-posed integral equation.

To illustrate the efficiency of Tikhonov-Morozov algorithm we consider the same equation as in Section 2.1:

$$Kx := \int_a^b k(t,s)x(s)ds = f(t), \quad [a,b] = [c,d] = [0,1],$$

$$\text{with } k(t,s) = e^{ts}, \quad K : L^2[a,b] \to L^2[c,d] \quad \text{and} \quad f_e(t) = \frac{e^{t+1}-1}{t+1}.$$

Note that in $L^2[a,b]$ the norm is generated by the following inner product:

$$\langle [f],[g] \rangle_{L^2} := \int_a^b f(t)\bar{g}(t)dt, \quad f \in [f], \quad g \in [g],$$

where $[f],[g]$ are equivalence classes of functions, i.e., two functions are in the same class if they only differ on a set of Lebesgue's measure zero. Also, the number of partitions

we will use are $n = 4, 8, 16, 32$ while varying $t \in [0, 1]$ for our tables. Note that K is self-adjoint, that is $K^* = K$. Indeed, since $k(t, s) = e^{ts}$, K^* would lead to the following,

$$K^*x := \int_c^d \overline{k(s,t)}x(t)dt, \text{ where } [c, d] = [a, b] = [0, 1].$$

Then $\overline{k(s,t)} = \overline{e^{st}}$. Since t and s are real valued, $\overline{e^{st}} = e^{ts}$. Therefore, $K^* = K$.

Table 4.1. Comparison of Errors

$\|x_e - x_{\alpha,\delta}\|,\ \delta_n = \frac{e^2h^2}{3} + \sigma$			
$n\backslash\sigma$	0.0001	0.05	0.1
4	0.0876	0.1177	0.1436
8	0.0298	0.0376	0.0406
16	0.0052	0.0273	0.0373
32	0.0005	0.0196	0.0338

Table 4.2. Values of α by Discrepancy Principle

σ	$n = 4$	$n = 8$	$n = 16$	$n = 32$
0.1	0.173360	0.068038	0.032997	0.018896
0.05	0.144455	0.050745	0.022335	0.011284
0.0001	0.109950	0.025159	0.005009	0.001159

We discretize the operator K by again using the trapezoidal rule to approximate the integral

$$(K_n x)_i := h\left[\frac{1}{2}x_0 + \frac{1}{2}e^{ih}x_n + \sum_{j=1}^{n-1} e^{ijh^2}x_j\right] = (f_\delta)_i,$$

where x_j is an approximate value for $x(jh),\ j = 0, ..., n,$ and $(f_\delta)_i = f(ih),\ i = 0, ..., n,$ is a noisy right-hand side. Then for the discrete analog of the Tikhonov solution, one

gets

$$x_{\alpha,\delta} = (\alpha I_n + K_n^* K_n)^{-1} K_n^* f_\delta, \quad K_n^* = K_n.$$

In order to implement the discrepancy principle, one has to solve the following nonlinear equation:

$$\varphi(\alpha) := \|K_n x_{\alpha,\delta} - f_\delta\|^2 - \delta^2 = 0.$$

Introduce the notation of

$$G_n := (\alpha I_n + K_n^* K_n)^{-1}.$$

Substituting $x_{\alpha,\delta} = G_n K_n^* f_\delta$ in the expression for $\varphi(\alpha)$, one concludes

$$\varphi(\alpha) := \|K_n G_n K_n^* f_\delta - f_\delta\|^2 - \delta^2 := \|(K_n G_n K_n^* - I_n) f_\delta\|^2 - \delta^2 = 0.$$

Table 4.3. Comparative Results for $\sigma = 0.0001$

	$x_e - x_{\alpha,\delta}$				
t	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$x_e = exp(t)$
0	-0.1852	-0.1845	-0.0803	-0.0134	1.00
0.25	-0.0912	-0.1220	-0.0616	-0.0193	1.28
0.50	0.0444	-0.0262	-0.0235	-0.0138	1.65
0.75	0.2355	0.1148	0.0424	0.0094	2.12
1.00	0.5003	0.3169	0.1476	0.0589	2.72

We solve $\varphi(\alpha) = 0$ by Newton's method:

$$\alpha_{j+1} = \alpha_j - \frac{\varphi(\alpha_j)}{\varphi'(\alpha_j)}, \quad j = 0, 1, \dots .$$

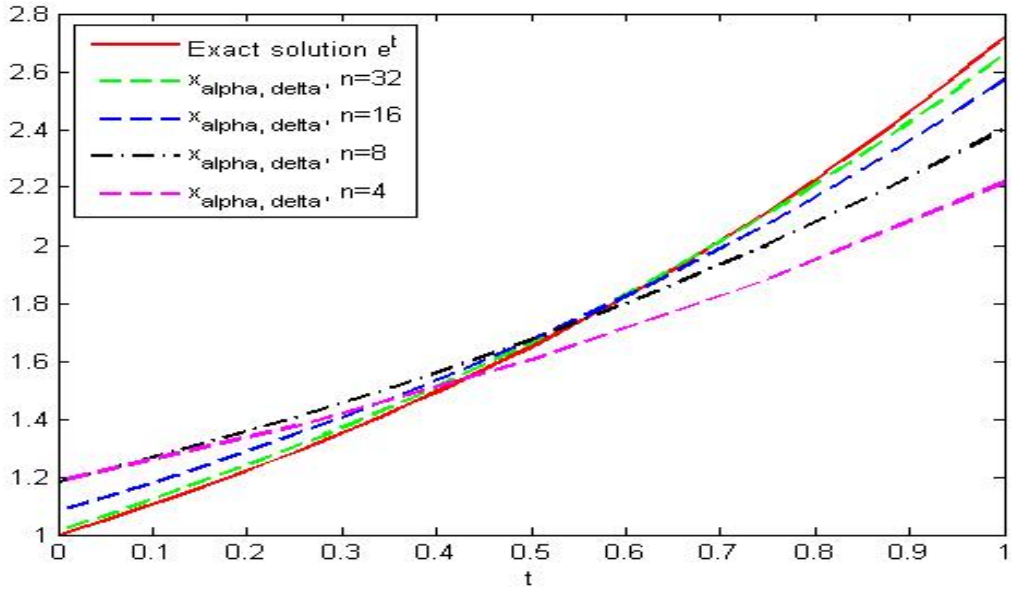

Figure 4.1. Exact and Approximate Solutions for $\sigma = 0.0001$

As it has been shown in the previous section,

$$\varphi'(\alpha) = 2\left\langle K_n \frac{dx_{\alpha,\delta}}{d\alpha},\ K_n x_{\alpha,\delta} - f_\delta \right\rangle.$$

Recall, that the operator in our example is real-valued. Besides,

$$\frac{dx_{\alpha,\delta}}{d\alpha} = -G_n x_{\alpha,\delta} = -G_n^2 K_n^* f_\delta.$$

Hence,

$$\varphi'(\alpha) = -2\left\langle K_n G_n^2 K_n^* f_\delta, K_n G_n K_n^* f_\delta - f_\delta \right\rangle = -2\left\langle K_n G_n^2 K_n f_\delta, (K_n G_n K_n - I_n) f_\delta \right\rangle,$$

since $K_n^* = K_n$. We terminate Newton's method when

$$|\alpha_{j+1} - \alpha_j| < 10^{-6} \text{ and } |\varphi(\alpha_j)| < 10^{-6},$$

which is comparable to the Matlab solver 'fsolve'.

Table 4.4. Comparative Results for $\sigma = 0.05$

	$x_e - x_{\alpha,\delta}$				
t	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$x_e = exp(t)$
0	-0.1980	-0.1868	-0.2004	-0.1838	1.00
0.25	-0.0880	-0.1153	-0.1344	-0.1274	1.28
0.50	0.0678	-0.0080	-0.0338	-0.0396	1.65
0.75	0.2845	0.1479	0.1139	0.0917	2.12
1.00	0.5816	0.3691	0.3254	0.2826	2.72

4.2 Sources of Noise

Table 4.5. Comparative Results for $\sigma = 0.1$

	$x_e - x_{\alpha,\delta}$				
t	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$x_e = exp(t)$
0	-0.2027	-0.1776	-0.2302	-0.2429	1.00
0.25	-0.0815	-0.1040	-0.1502	-0.1634	1.28
0.50	0.0888	0.0061	-0.0312	-0.0449	1.65
0.75	0.3236	0.1653	0.1407	0.1266	2.12
1.00	0.6435	0.3906	0.3835	0.3698	2.72

To generate f_δ, we add a noise term to the exact function f_e. That noise term

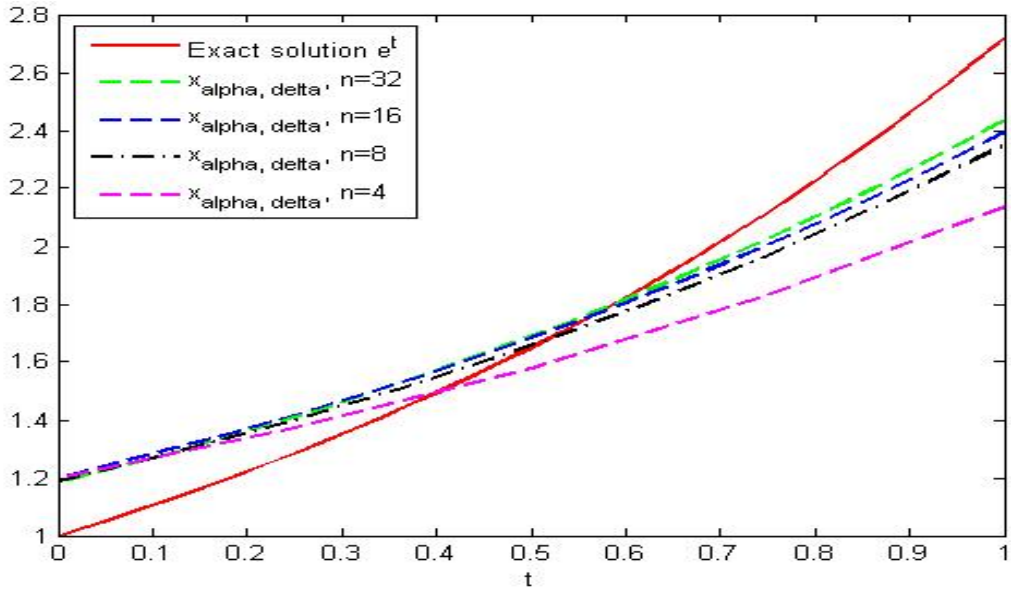

Figure 4.2. Exact and Approximate Solutions for $\sigma = 0.05$

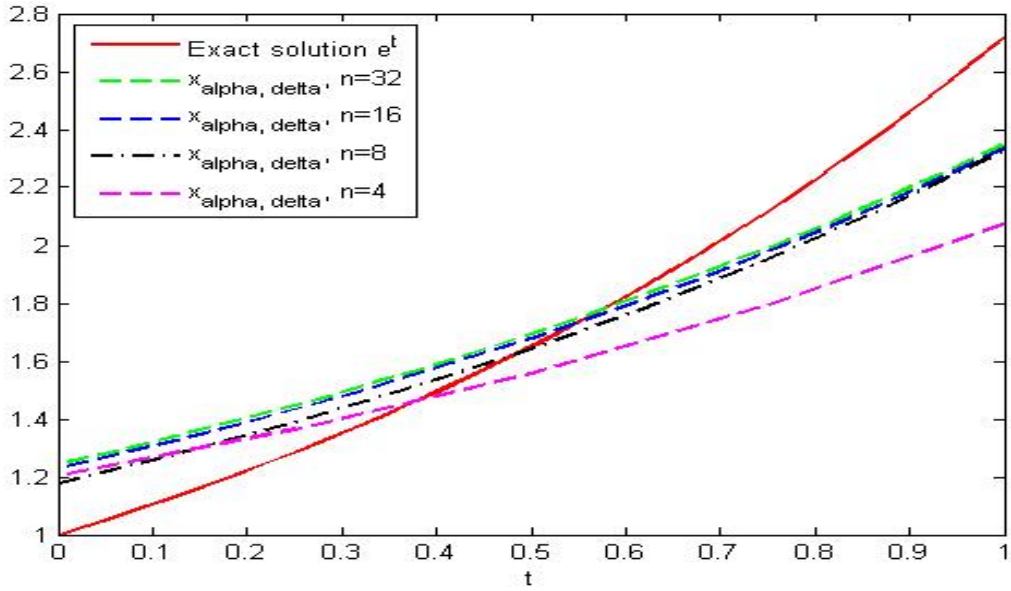

Figure 4.3. Exact and Approximate Solutions for $\sigma = 0.1$

includes an input value of σ, and is equal to $\sigma\sqrt{2}\sin(10\pi t)$. Therefore, f_δ is:

$$f_\delta = f_e + \sigma\sqrt{2}\sin(10\pi t),$$

where $f_e = \frac{e^{t+1}-1}{t+1}$, see Figure 4.5. In our numerical simulations, the error in the right-hand side comes from two sources. First, it is the error due to inaccurate measurements, which we assume to be $\sigma\sqrt{2}\sin(10\pi t)$. Second, it is the discretization error that is generated when the integral is replaced with a trapezoidal formula.

For large values of n, the discretization error can be ignored. However, we want to compare our results to those obtained by 'naive' numerical method used in section 2.1. There we used $n = 4, 8, 16$, and 32, since for larger values of n, the matrix K_n becomes singular to the MatLab precision. Hence, in this experiment, we take the same values of n (4, 8, 16, and 32), and therefore the discretization error must be thoroughly investigated. Denote the exact right-hand side for each n by f_{e_n}. One has

$$\|f_{e_n} - f_\delta\| \leq \|f_{e_n} - f_e\| + \|f_e - f_\delta\|.$$

The error for the composite trapezoidal rule

$$f_{e_n} - f_e = \frac{g''(\xi)(b-a)}{12}h^2,$$

where $\xi \in [a, b]$. In our case,

$$g(s) \approx e^{ts}e^s = e^{s(t+1)} \text{ and } [a, b] = [0, 1].$$

Thus,

$$\|f_{e_n} - f_e\| = \left(\int_0^1 \frac{(t+1)^4 e^{2(t+1)} h^4}{12^2} dt\right)^{\frac{1}{2}} \leq \frac{h^2}{12}\left(\int_0^1 2^4 e^4 dt\right)^{\frac{1}{2}} = \frac{e^2 h^2}{3}, \text{ since } t \in [0,1].$$

One can also see that

$$\|f_e - f_\delta\|^2 = 2\int_0^1 \sigma^2 \sin^2(10\pi t) dt = 2\sigma^2 \int_0^1 \frac{1-\cos(20\pi t)}{2} dt = 2\sigma^2 \frac{1}{2} = \sigma^2.$$

Hence, $\|f_{e_n} - f_\delta\| \leq \frac{e^2 h^2}{3} + \sigma := \delta_n$. Note that δ_n changes as n changes since h, the step size, is dependent on n, the number of partitions. We use the L^2-norm instead of the infinity-norm since the discrepancy principle of Morozov is justified for Hilbert spaces. The discrete analog of this norm is used to compute $\varphi(\alpha)$ in the Newton method for the same reason.

4.3 Simulation

For the experiment, we apply the discrepancy principle with $\sigma = 0.1$, $\sigma = 0.05$, and $\sigma = 0.0001$. As α_0, the initial approximation, we take $\frac{1}{4}$, because $\varphi(0) < 0$ and $\varphi(\frac{1}{2}) > 0$. The first Table 4.1 shows a comparison of the norms and confirms that as n gets larger we have convergence to the exact solution for x_e.

The second Table 4.2, shows α versus σ for different values of n. Note that $\sigma = 0$ is not included in the table since we would only have the discretization error of the trapezoidal rule (as done in Table 2.1). One can see in the Table 4.2, as the noise gets smaller, α also gets smaller, as to be expected. After evaluating α by Morozov's discrepancy principle, we can now find $x_{\alpha,\delta}$ and compare it to our exact value of $x_e = e^t$ for $\sigma = 0.1,\ 0.05,\ 0.0001$, and $n = 4, 8, 16, 32$. Table 4.3 shows how $x_{\alpha,\delta}$ differs from

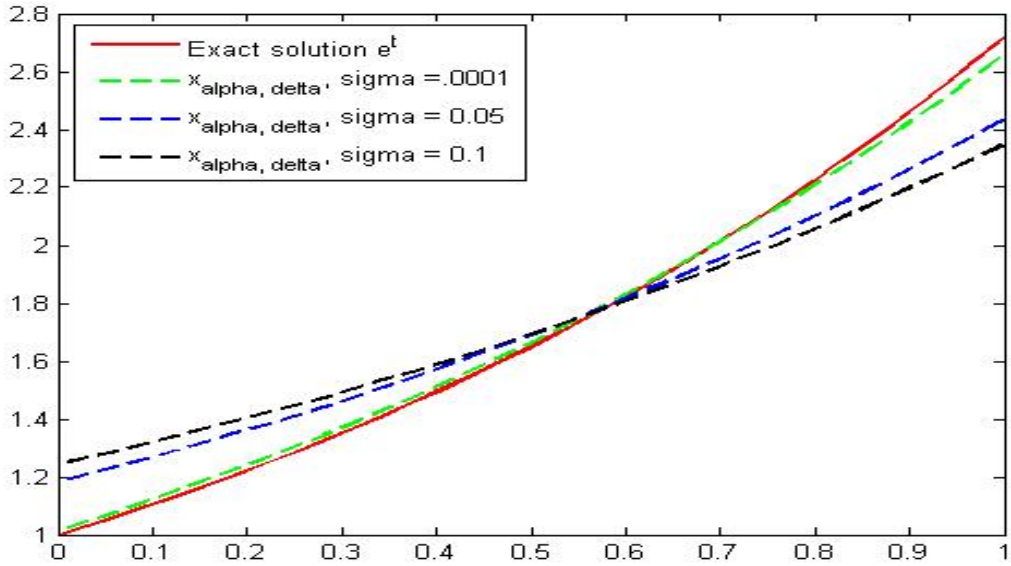

Figure 4.4. Exact and Approximate Solutions for $n = 32$

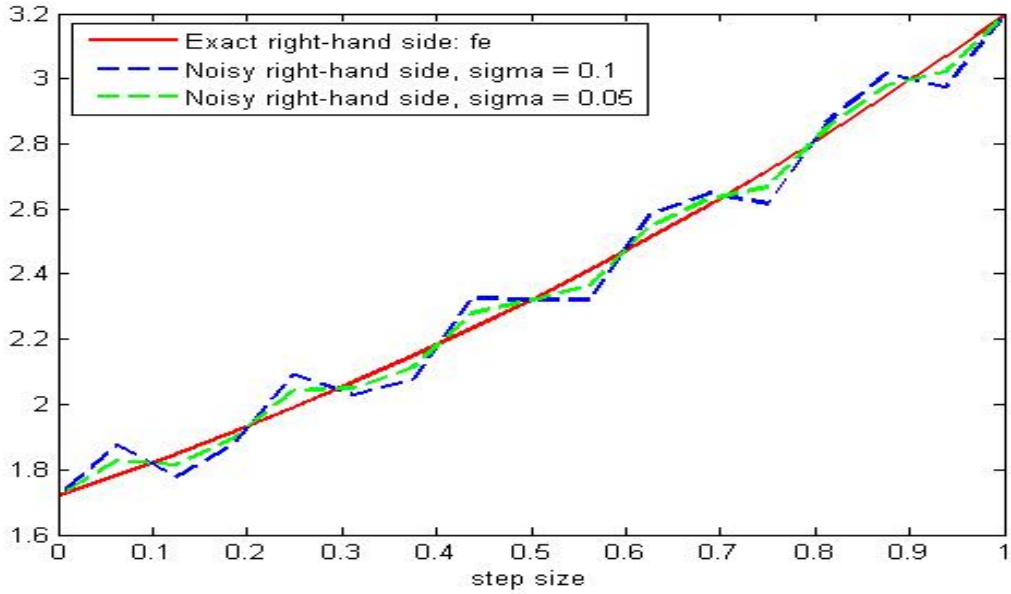

Figure 4.5. Exact and Approximate Values of Right-hand Side

x_e for $\sigma = 0.0001$, our smallest value for σ, Figure 4.1 is the corresponding picture. As you can see, as n gets larger, the approximation converges towards the exact solution as guaranteed by the convergence theorem. Table 4.4 and Figure 4.2 show a similar result for $\sigma = 0.05$, as does Table 4.5 and Figure 4.3 for $\sigma = 0.1$, our largest σ. One can see that when σ is small we get a much better approximation of our exact solution. The final Figure 4.4 compares the exact and approximate solutions for various σ and $n = 32$. Again when $n = 32$ and $\sigma = 0.0001$ the approximation is the best.

In conclusion, this part of the experiment worked as the general theory said. We did have to use careful consideration of the error term since our n was so small. The general theory tends toward a larger n and therefore does not consider what an important role this term can play for a very small number of partitions such as we used. The next chapter will compare our regularized version of the integral to our naive discretization used in chapter 2.

Chapter 5

DISCUSSION

5.1 Summary

For our summary we are using $\sigma = 0.05$ as part of the noise in the Tikhonov regularization with Morozov's discrepancy since it is the middle of the three that we choose to work with. We start by comparing the two tables, Table 5.1 for our 'naive' discretization and Table 5.2 for the Tikhonov-Morozov algorithm. Note that for the 'naive' discretization, $\sigma = 0$, and the only noise is due to finite-dimensional approximatation for the integral. As one can see, the numerical solutions obtained by the Tikhonov-Morozov algorithm are much better than those done by the 'naive' discretization. The errors for each n show that while the 'naive' discretization oscillates wildly and gets worse as n increases, the Tikhonov-Morozov algorithm actually gets better as n gets larger. This is confirmed by the general theory.

We now move to the figures. Again Figure 5.1 refers to the 'naive' discretization while Figure 5.2 refers to the Tikhonov-Morozov algorithm. As foretold in the tables, the 'naive' discretization varies wildly while the Tikhonov-Morozov algorithm generates a smooth line that converges toward the exact value as n gets larger. All of the values of n could not even be shown in the 'naive' discretization due to the fact that at $n = 16$ and $n = 32$ the matrix is nearly singular and does not return a better result but a much

Table 5.1. Results for the 'Naive' Discretization

	$x(t_i) - x_i$				
t	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$x(t_i) = exp(t_i)$
0	0.56	0.53	-3.20	-31.14	1.00
0.25	1.95	-0.74	45.30	13.40	1.28
0.50	0.69	-3.11	24.17	-12.32	1.65
0.75	3.13	-0.95	66.94	33.04	2.12
1.0	1.63	1.49	-0.72	-9.83	2.72

Table 5.2. Comparative Results for $\sigma = 0.05$

	$x_e - x_{\alpha,\delta}$				
t	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$x_e = exp(t)$
0	-0.1980	-0.1868	-0.2004	-0.1838	1.00
0.25	-0.0880	-0.1153	-0.1344	-0.1274	1.28
0.50	0.0678	-0.0080	-0.0338	-0.0396	1.65
0.75	0.2845	0.1479	0.1139	0.0917	2.12
1.00	0.5816	0.3691	0.3254	0.2826	2.72

worse one. Note that even thought we are showing $\sigma = 0.05$ as our representative for Tikhonov-Morozov algorithm, the one with $\sigma = 0.1$, our largest value for σ, is still much better approximation than the naive discretization, which uses $\sigma = 0$.

5.2 Advantages

There are definite advantages to using the Tikhonov regularization with Morozov's discrepancy principle. The first and most obvious is that it returns a much better approximate solution than the 'naive' discretization does. While using the same trapezoidal rule for the approximation of the integral, the relatively simple regularization by the Tikhonov method along with Morozov's discrepancy principle produces little to no discernible error

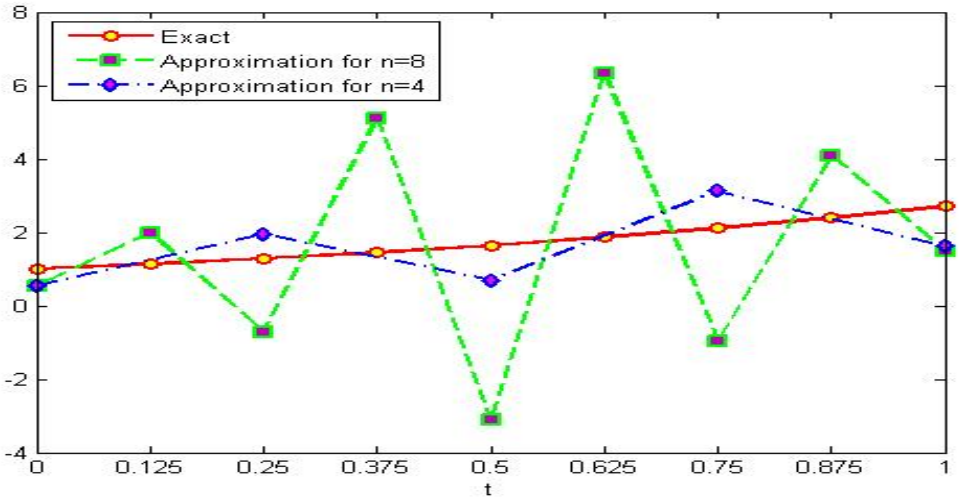

Figure 5.1. Exact and Approximate Solutions

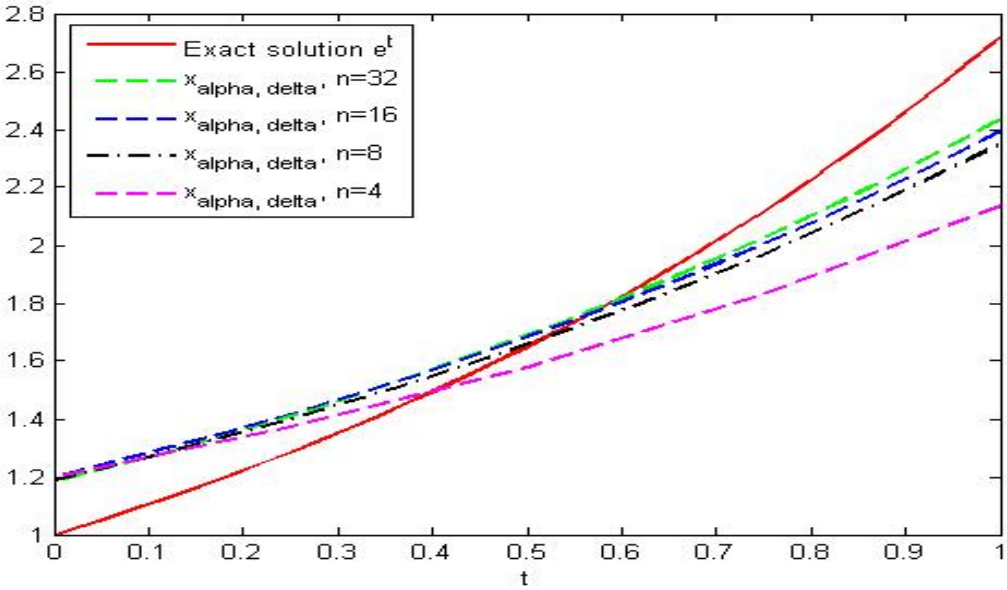

Figure 5.2. Exact and Approximate Solution for $\sigma = 0.05$

as the partitions get sufficiently large, meaning you get a much closer approximation as the number of partitions gets larger for any amount of noise. In other words, the accuracy of the regularized solutions gets better as n approaches infinity, which is not the case for the 'naive' discretization.

5.3 Disadvantages

The biggest disadvantage of Tikhonov regularization with Morozov's discrepancy principle is the repeated matrix manipulation done to compute the solution. Since the approximate solution uses the inverse of $(\alpha_j I + K^* K)$ one has to make sure that it exists for every j and to start Newton's iterations with an overestimate of α rather than underestimate.

Another disadvantage of Tikhonov regularization is the over-smoothing effect. In order to reconstruct nonsmooth or discontinuous solutions, one has to use a different penalty term.

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APPENDIX A

```
% This program is written by MaryGeorge Whitney
%
% This program solves a linear system for an inverse problem
% by using the trapizoidal rule to approximate the integral
%
% The integral is e^ts x(s) ds = f(t)
% Where f(t) = (e^(t+1) - 1)/(t+1)
% The exact value of x = exp(t)
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%             Variable List
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% K - matrix for the integral answers
% B - array matrix for the approximation for chart
% check - while loop variable
% ck - while loop variable
% E - array matrix for the exact values
% m - inline function for the right hand side of equation
% g - inline finction for the exact value of x
% h - subinterval = 1/n
% i - looping variable for the t partition
% j - looping variable for the s partition
% n - number of subintervals - input by user
% N - the number of subintervals plus 1 for matrix reference
% R - array matrix for error
% s - partition variable = j*h
% t - partition variable = i*h
% w - input variable for repeating program
% x - array matrix for the approximated answer of K\f
% f - array matrix for the solution answers
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%
%               Main Program
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clc;
check = 0;
while check == 0

    % Introduction and input of number of intervals
    fprintf('This program solves a liner system for an inverse problem.\n');
    fprintf('It uses the trapizoidal rule to approximate the integral:\n');
    fprintf('e^ts x(s) ds = f(t) the integral goes from 0 to 1\n');
    fprintf('Please enter the number of partitions desired:\n');
    n = input(' ');

    % Initalize variables
    ck = 0;
    h = 1/n;
    N = n + 1;
    K = zeros(N,N);
    f = zeros(N,1);
    E = zeros(N,1);
    x = zeros(N,1);
    B = zeros(N,1);
    R = zeros(N,1);
    m = inline ('(exp(t+1) - 1)/(t+1)');
    g = inline ('exp (t)');

    % Build matrices
    for i=1:N
        t = (i-1)*h;
        f(i) = m(t);
        E(i) = g(t);
        for j=1:N
            s = (j-1)*h;
            K(i,j) = exp(t*s);
        end % ends j loop
    end % ends i loop

    %Finish off the K matrix
```

```
K(:,1) = K(:,1)/2;
K(:,N) = K(:,N)/2;
K = h*K;

% Build solution
x = K\f;

% Build matrix for print chart
for i=1:N
    t = (i-1)*h;
    switch t
        case 0
            B(i) = x(i);
            R(i) = E(i) - x(i);
        case .25
            B(i) = x(i);
            R(i) = E(i) - x(i);
        case .5
            B(i) = x(i);
            R(i) = E(i) - x(i);
        case .75
            B(i) = x(i);
            R(i) = E(i) - x(i);
        case 1
            B(i) = x(i);
            R(i) = E(i) - x(i);
        otherwise
            B(i) = 0;
            R(i) = 0;
    end % ends switch
end % ends for i

%Print selected solutions
fprintf('Remember that the number of partitions is: %3.0f \n', n);
for i=1:N
    if B(i) ~= 0
        t = (i-1)*h;
        fprintf('t = %3.2f\n', t);
        fprintf('Our approximation is %5.2f\n', B(i));
        fprintf('Our exact value is %5.2f\n', E(i));
        fprintf('With an error of %5.2f.\n\n',R(i));
```

```
        end % ends if
    end % ends for

    %Plotting the compairison
    i=1:N;
    q=(i-1)/n;
    plot(q,g(q),'ro-',q,x(i),'bd:');
    axis auto;
    title('Exact and approximate functions');
    xlabel('step size');
    legend('Exact','Approximation','Location','NorthWest');

    while ck == 0
        fprintf('Would you like to run again?\n');
        fprintf('Please type ''y'' or ''n''.\n');
        w = input(' ','s');
        fprintf('\n\n\n');
        switch lower(w)
            case 'y'
                check = 0;
                ck = 1;
            case 'n'
                check = 1;
                ck = 1;
            otherwise
                fprintf('Please try again');
                check = 0;
                ck = 0;
        end % ends switch
    end % ends while ck loop
end % ends while check loop
```

APPENDIX B

```
Exercise1
This program solves a liner system for an inverse problem.
It uses the trapizoidal rule to approximate the integral:
e^ts x(s) ds = f(t) the integral goes from 0 to 1
Please enter the number of partitions desired:
 4
Remember that the number of partitions is:   4
t = 0.00
Our approximation is  0.56
Our exact value is  1.00
With an error of  0.44.

t = 0.25
Our approximation is  1.95
Our exact value is  1.28
With an error of -0.67.

t = 0.50
Our approximation is  0.69
Our exact value is  1.65
With an error of  0.95.

t = 0.75
Our approximation is  3.13
Our exact value is  2.12
With an error of -1.02.

t = 1.00
Our approximation is  1.63
Our exact value is  2.72
With an error of  1.09.

Would you like to run again?
Please type 'y' or 'n'.
 y
```

```
This program solves a liner system for an inverse problem.
It uses the trapizoidal rule to approximate the integral:
e^ts x(s) ds = f(t) the integral goes from 0 to 1
Please enter the number of partitions desired:
 8
Remember that the number of partitions is:   8
t = 0.00
Our approximation is  0.53
Our exact value is  1.00
With an error of  0.47.

t = 0.25
Our approximation is -0.74
Our exact value is  1.28
With an error of  2.02.

t = 0.50
Our approximation is -3.11
Our exact value is  1.65
With an error of  4.76.

t = 0.75
Our approximation is -0.95
Our exact value is  2.12
With an error of  3.07.

t = 1.00
Our approximation is  1.49
Our exact value is  2.72
With an error of  1.23.

Would you like to run again?
Please type 'y' or 'n'.
 y
```

```
This program solves a liner system for an inverse problem.
It uses the trapizoidal rule to approximate the integral:
e^ts x(s) ds = f(t) the integral goes from 0 to 1
Please enter the number of partitions desired:
 16
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 6.750684e-019.
> In <a href="error:G:\Mathematics\Dr. Smirnova\MyThesis\
Exercise1.m,82,1">Exercise1 at 82</a>
Remember that the number of partitions is:  16
t = 0.00
Our approximation is -3.20
Our exact value is  1.00
With an error of  4.20.

t = 0.25
Our approximation is 45.30
Our exact value is  1.28
With an error of -44.01.

t = 0.50
Our approximation is 24.17
Our exact value is  1.65
With an error of -22.52.

t = 0.75
Our approximation is 66.94
Our exact value is  2.12
With an error of -64.82.

t = 1.00
Our approximation is -0.72
Our exact value is  2.72
With an error of  3.44.

Would you like to run again?
Please type 'y' or 'n'.
 y
```

```
This program solves a liner system for an inverse problem.
It uses the trapizoidal rule to approximate the integral:
e^ts x(s) ds = f(t) the integral goes from 0 to 1
Please enter the number of partitions desired:
 32
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 2.697308e-019.
> In <a href="error:G:\Mathematics\Dr. Smirnova\MyThesis
\Exercise1.m,82,1">Exercise1 at 82</a>
Remember that the number of partitions is:  32
t = 0.00
Our approximation is -31.14
Our exact value is  1.00
With an error of 32.14.

t = 0.25
Our approximation is 13.04
Our exact value is  1.28
With an error of -11.76.

t = 0.50
Our approximation is -12.32
Our exact value is  1.65
With an error of 13.97.

t = 0.75
Our approximation is 33.04
Our exact value is  2.12
With an error of -30.92.

t = 1.00
Our approximation is -9.83
Our exact value is  2.72
With an error of 12.55.

Would you like to run again?
Please type 'y' or 'n'.
 n
```

APPENDIX C

```
% This program was written by MaryGeorge L. Whitney
% It is part of a master thesis project under the direction of
% Dr. Alexandria Smirnova
%
% This program uses Newton's method to produce a regularization
% strategy for an ill-posed integral problem
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%                    Variable List
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% alpha0 - inital value (approximation) for alpha used in regularization
% alpha1 - scalar for regularization
% alphaTol - tolerance used for alpha
% cek - while loop variable
% check - while loop variable
% choice - input variable for continuation
% ck - while loop variable
% sigma - input variable, noise, noted as sigma in the text
% fe - array variable, right-hand side exact
% fdelta - array variable, noisy right-hand side
% g - inline function, calculates exact x
% h - step size
% i - loop variable
% I - matrix variable, identity matrix
% j - loop variable
% K - matrix variable, integral from our equation
% Kstar - matrix variable, Hermitian matrix of K
% m - inline function, calculates right-hand side (fe)
% n - input variable, number of partitions
% N - number of partitions plus one for matrices
% phi - regularization equation based on alpha, return value from Newton's
```

```
% phiprime - derivative of phi, return value from Newton's
% phiTol - used to check tolerance of phi
% q - plot variable
% R - array variable, error difference for xe and xalphadelta
% s - partition variable j*h
% t - partition variable i*h
% tol - tolerance allowed
% w - inline function for noisy right-hand side (fdelta)
% xalphadelta - array variable, regualized noisy answer to Kx=f
% xexact - array variable, exact answer to Kx=f
% G - matrix variable, used for alpha calculation (alpha*I+K*K)^-1
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%                    Main Program
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clc;
check = 0;

while check == 0 %loop for the program

cek = 0;
% Input noise: sigma and number of partitions: n
while cek == 0
    fprintf('Please enter the amount of noise.\n');
    fprintf('This is sigma in our problem.\n');
    fprintf('Sigma must be > 0 and < 1.\n');
    sigma = input(' ');
    fprintf('\n');
    if (sigma < 0) | (sigma > 1)
        fprintf('Please try again. Sigma is out of range.\n');
    else
        cek = 1;
    end % Ends sigma range check
end % ends while loop for delta

fprintf('Please enter the number of partitions.\n');
n = input(' ');
fprintf('\n \n \n');
```

```
% Initialize variables and inline functions
N = n+1;
h = 1/n;
I = eye(N);
t = 0;
fe = zeros(N,1);
fdelta = zeros(N,1);
xexact = zeros(N,1);
s = 0;
K = zeros(N,N);
Kstar = zeros(N,N);
alpha0 = .25; %inital approximation for alpha
alphaTol = 1; %inital tolerance for alpha
phiTol = 1; %inital tolerance for phi
tol = 10^(-6); % tolerance level
G = zeros(N,N);
phi = 0;
phiprime = 0;
alpha1 = 0;
xalphadelta = zeros(N,1);
R = zeros(N,1);
q = 0;
m = inline('(exp(t+1)-1)/(t+1)'); %inline function for f-exact
g = inline('exp(t)'); %inline function for exact x
w = inline('(exp(t+1)-1)/(t+1) + sigma *sqrt(2)* sin(10*pi*t)', 'sigma', 't'); %function

% Build matrix K
%Trapezoidal rule for integral
for i = 1:N
    t = (i-1)*h;
    fe(i) = m(t);
    fdelta(i) = w(sigma,t);
    xexact(i) = g(t);
    for j = 1:N
        s = (j-1)*h;
        K(i,j) = exp(t*s);
    end %end for j loop
end %end for i loop
K(:,1) = K(:,1)/2;
K(:,N) = K(:,N)/2;
```

```
K = h*K;
Kstar = K; %used strictly for notation

%Build alpha using Newton's method
count = 0;
while (alphaTol > tol | phiTol > tol) & count < 500
    G = inv(alpha0*I + Kstar*K);
    [phi, phiprime] = gnew(K,G,Kstar,fdelta,sigma,h);
    alpha1 = alpha0 - phi/phiprime;
    alphaTol = abs(alpha1 - alpha0);
    phiTol = abs(phi);
    alpha0 = alpha1;
    count = count + 1;
end %end while loop for Newton's alpha

fprintf('Iteration count = %4d \n', count);
fprintf('Alpha tolerance = %12.3e \n', alphaTol);
fprintf('Phi tolerance = %12.3e \n', phiTol);

%Build x(alpha, delta)
xalphadelta = G*Kstar*fdelta;

%Build error matrix for tables
for i=1:N
    t = (i-1)*h;
    switch t
        case 0
            R(i) = xexact(i) - xalphadelta(i);
        case .25
            R(i) = xexact(i) - xalphadelta(i);
        case .5
            R(i) = xexact(i) - xalphadelta(i);
        case .75
            R(i) = xexact(i) - xalphadelta(i);
        case 1
            R(i) = xexact(i) - xalphadelta(i);
        otherwise
            R(i) = 0;
    end % ends switch
end % ends for i
```

```
%Print selected solutions
fprintf('Our sigma is: %5.4f \n', sigma);
fprintf('The number of partitions is: %3.0f \n', n);
for i=1:N
    if R(i) ~= 0
        t = (i-1)*h;
        fprintf('t = %3.2f \n', t);
        fprintf('alpha = %7.6f \n', alpha0);
        fprintf('Our approximation is %5.4f\n', xalphadelta(i));
        fprintf('Our exact value is %5.4f\n', xexact(i));
        fprintf('With an error of %5.4f.\n\n',R(i));
    end % ends if
end % ends for

%Plotting the compairison
i=1:N;
q=(i-1)*h;
plot(q,xexact(i),'ro-',q,xalphadelta(i),'bd:');
axis auto;
title('Exact and approximate functions');
xlabel('step size');
legend('Exact solution xexact(i)','Approximate solution xalphadelta(i)','Location','Nort

ck = 0;
while ck == 0
    fprintf('Would you like to run again?\n');
    fprintf('Please type ''y'' or ''n''.\n');
    choice = input(' ','s');
    fprintf('\n\n\n');
    switch lower(choice)
        case 'y'
            check = 0;
            ck = 1;
        case 'n'
            check = 1;
            ck = 1;
        otherwise
            fprintf('Please try again');
            check = 0;
            ck = 0;
    end % ends switch
```

```
end % ends while loop for another run

end % ends while for program loop
```

APPENDIX D

```
function [y1, y2] = gnew(K,G,Kstar,fdelta,sigma,h);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This program is written by MaryGeorge Whitney
% It is the Newton method for Phi and Phiprime
% It uses a special norm for compact Hilbert spaces
%
% Inputs: K - integral matrix
%         Kstar - equal to K - used as notation
%         G - inverse matrix for (alpha * I + Kstar * K)
%         fdelta - noisy right-hand side
%         sigma - input noise
%         h - tep size
%
%Outputs: Phi at alpha
%         Phiprime at alpha
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Calculate y1 that is phi
z = K*G*Kstar*fdelta - fdelta;
y1 = (z'*z)*h - (sigma00 + exp(2) * (h^2)/3)^2;

%Calculate y2 that is phiprime
a=-K*G*G*Kstar*fdelta;
b=K*G*Kstar*fdelta-fdelta;
y2 = 2*a'*b;
```

APPENDIX E

```
Exer2
Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .1

Please enter the number of partitions.
 4

Iteration count =   39
Alpha tolerance =   4.062e-007
Phi tolerance =   9.396e-007
Our sigma is: 0.1000
The number of partitions is:   4
t = 0.00
alpha = 0.173860
Our approximation is 1.2027
Our exact value is 1.0000
With an error of -0.2027.

t = 0.25
alpha = 0.173860
Our approximation is 1.3655
Our exact value is 1.2840
With an error of -0.0815.

t = 0.50
alpha = 0.173860
Our approximation is 1.5600
Our exact value is 1.6487
With an error of 0.0888.

t = 0.75
alpha = 0.173860
```

```
Our approximation is 1.7934
Our exact value is 2.1170
With an error of 0.3236.

t = 1.00
alpha = 0.173860
Our approximation is 2.0748
Our exact value is 2.7183
With an error of 0.6435.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .1

Please enter the number of partitions.
 8

Iteration count =   87
Alpha tolerance =   4.903e-007
Phi tolerance =   9.619e-007
Our sigma is: 0.1000
The number of partitions is:   8
t = 0.00
alpha = 0.068038
Our approximation is 1.1776
Our exact value is 1.0000
With an error of -0.1776.

t = 0.25
alpha = 0.068038
Our approximation is 1.3880
Our exact value is 1.2840
With an error of -0.1040.
```

```
t = 0.50
alpha = 0.068038
Our approximation is 1.6426
Our exact value is 1.6487
With an error of 0.0061.

t = 0.75
alpha = 0.068038
Our approximation is 1.9517
Our exact value is 2.1170
With an error of 0.1653.

t = 1.00
alpha = 0.068038
Our approximation is 2.3277
Our exact value is 2.7183
With an error of 0.3906.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .1

Please enter the number of partitions.
 16

Iteration count =  179
Alpha tolerance =   4.998e-007
Phi tolerance =   9.491e-007
Our sigma is: 0.1000
The number of partitions is:  16
t = 0.00
alpha = 0.032997
Our approximation is 1.2302
Our exact value is 1.0000
```

```
With an error of -0.2302.

t = 0.25
alpha = 0.032997
Our approximation is 1.4342
Our exact value is 1.2840
With an error of -0.1502.

t = 0.50
alpha = 0.032997
Our approximation is 1.6799
Our exact value is 1.6487
With an error of -0.0312.

t = 0.75
alpha = 0.032997
Our approximation is 1.9763
Our exact value is 2.1170
With an error of 0.1407.

t = 1.00
alpha = 0.032997
Our approximation is 2.3348
Our exact value is 2.7183
With an error of 0.3835.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .1

Please enter the number of partitions.
 32

Iteration count =  360
```

```
Alpha tolerance =   4.622e-007
Phi tolerance =   9.974e-007
Our sigma is: 0.1000
The number of partitions is:  32
t = 0.00
alpha = 0.018896
Our approximation is 1.2429
Our exact value is 1.0000
With an error of -0.2429.

t = 0.25
alpha = 0.018896
Our approximation is 1.4474
Our exact value is 1.2840
With an error of -0.1634.

t = 0.50
alpha = 0.018896
Our approximation is 1.6936
Our exact value is 1.6487
With an error of -0.0449.

t = 0.75
alpha = 0.018896
Our approximation is 1.9904
Our exact value is 2.1170
With an error of 0.1266.

t = 1.00
alpha = 0.018896
Our approximation is 2.3485
Our exact value is 2.7183
With an error of 0.3698.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
```

```
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .05

Please enter the number of partitions.
 4

Iteration count =   40
Alpha tolerance =   4.719e-007
Phi tolerance =   9.466e-007
Our sigma is: 0.0500
The number of partitions is:   4
t = 0.00
alpha = 0.144455
Our approximation is 1.1980
Our exact value is 1.0000
With an error of -0.1980.

t = 0.25
alpha = 0.144455
Our approximation is 1.3721
Our exact value is 1.2840
With an error of -0.0880.

t = 0.50
alpha = 0.144455
Our approximation is 1.5809
Our exact value is 1.6487
With an error of 0.0678.

t = 0.75
alpha = 0.144455
Our approximation is 1.8325
Our exact value is 2.1170
With an error of 0.2845.

t = 1.00
alpha = 0.144455
Our approximation is 2.1367
Our exact value is 2.7183
With an error of 0.5816.
```

```
Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .05

Please enter the number of partitions.
 8

Iteration count =   88
Alpha tolerance =   5.879e-007
Phi tolerance =   8.928e-007
Our sigma is: 0.0500
The number of partitions is:   8
t = 0.00
alpha = 0.050745
Our approximation is 1.1868
Our exact value is 1.0000
With an error of -0.1868.

t = 0.25
alpha = 0.050745
Our approximation is 1.3993
Our exact value is 1.2840
With an error of -0.1153.

t = 0.50
alpha = 0.050745
Our approximation is 1.6567
Our exact value is 1.6487
With an error of -0.0080.

t = 0.75
alpha = 0.050745
Our approximation is 1.9691
```

```
Our exact value is 2.1170
With an error of 0.1479.

t = 1.00
alpha = 0.050745
Our approximation is 2.3492
Our exact value is 2.7183
With an error of 0.3691.

Would you like to run again?
Please type 'y' or 'n'.
 y




Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .05

Please enter the number of partitions.
 16

Iteration count =  179
Alpha tolerance =   7.119e-007
Phi tolerance =   9.724e-007
Our sigma is: 0.0500
The number of partitions is:  16
t = 0.00
alpha = 0.022335
Our approximation is 1.2004
Our exact value is 1.0000
With an error of -0.2004.

t = 0.25
alpha = 0.022335
Our approximation is 1.4184
Our exact value is 1.2840
With an error of -0.1344.
```

```
t = 0.50
alpha = 0.022335
Our approximation is 1.6825
Our exact value is 1.6487
With an error of -0.0338.

t = 0.75
alpha = 0.022335
Our approximation is 2.0031
Our exact value is 2.1170
With an error of 0.1139.

t = 1.00
alpha = 0.022335
Our approximation is 2.3928
Our exact value is 2.7183
With an error of 0.3254.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .05

Please enter the number of partitions.
 32

Iteration count =  361
Alpha tolerance =   6.733e-007
Phi tolerance =   9.790e-007
Our sigma is: 0.0500
The number of partitions is:  32
t = 0.00
alpha = 0.011284
Our approximation is 1.1838
Our exact value is 1.0000
```

```
With an error of -0.1838.

t = 0.25
alpha = 0.011284
Our approximation is 1.4115
Our exact value is 1.2840
With an error of -0.1274.

t = 0.50
alpha = 0.011284
Our approximation is 1.6884
Our exact value is 1.6487
With an error of -0.0396.

t = 0.75
alpha = 0.011284
Our approximation is 2.0253
Our exact value is 2.1170
With an error of 0.0917.

t = 1.00
alpha = 0.011284
Our approximation is 2.4357
Our exact value is 2.7183
With an error of 0.2826.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .0001

Please enter the number of partitions.
 4

Iteration count =   41
```

```
Alpha tolerance =   5.643e-007
Phi tolerance =   9.081e-007
Our sigma is: 0.0001
The number of partitions is:   4
t = 0.00
alpha = 0.109950
Our approximation is 1.1852
Our exact value is 1.0000
With an error of -0.1852.

t = 0.25
alpha = 0.109950
Our approximation is 1.3752
Our exact value is 1.2840
With an error of -0.0912.

t = 0.50
alpha = 0.109950
Our approximation is 1.6043
Our exact value is 1.6487
With an error of 0.0444.

t = 0.75
alpha = 0.109950
Our approximation is 1.8815
Our exact value is 2.1170
With an error of 0.2355.

t = 1.00
alpha = 0.109950
Our approximation is 2.2179
Our exact value is 2.7183
With an error of 0.5003.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
```

```
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .0001

Please enter the number of partitions.
 8

Iteration count =   90
Alpha tolerance =   8.963e-007
Phi tolerance =   7.416e-007
Our sigma is: 0.0001
The number of partitions is:   8
t = 0.00
alpha = 0.025159
Our approximation is 1.1845
Our exact value is 1.0000
With an error of -0.1845.

t = 0.25
alpha = 0.025159
Our approximation is 1.4060
Our exact value is 1.2840
With an error of -0.1220.

t = 0.50
alpha = 0.025159
Our approximation is 1.6749
Our exact value is 1.6487
With an error of -0.0262.

t = 0.75
alpha = 0.025159
Our approximation is 2.0022
Our exact value is 2.1170
With an error of 0.1148.

t = 1.00
alpha = 0.025159
Our approximation is 2.4014
Our exact value is 2.7183
With an error of 0.3169.
```

```
Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .0001

Please enter the number of partitions.
 16

Iteration count =  190
Alpha tolerance =   9.737e-007
Phi tolerance =   5.002e-007
Our sigma is: 0.0001
The number of partitions is:  16
t = 0.00
alpha = 0.005009
Our approximation is 1.0803
Our exact value is 1.0000
With an error of -0.0803.

t = 0.25
alpha = 0.005009
Our approximation is 1.3457
Our exact value is 1.2840
With an error of -0.0616.

t = 0.50
alpha = 0.005009
Our approximation is 1.6722
Our exact value is 1.6487
With an error of -0.0235.

t = 0.75
alpha = 0.005009
Our approximation is 2.0746
```

```
Our exact value is 2.1170
With an error of 0.0424.

t = 1.00
alpha = 0.005009
Our approximation is 2.5707
Our exact value is 2.7183
With an error of 0.1476.

Would you like to run again?
Please type 'y' or 'n'.
 y



Please enter the amount of noise.
This is sigma in our problem.
Sigma must be > 0 and < 1.
 .0001

Please enter the number of partitions.
 32

Iteration count =  395
Alpha tolerance =   9.989e-007
Phi tolerance =   3.427e-007
Our sigma is: 0.0001
The number of partitions is:  32
t = 0.00
alpha = 0.001159
Our approximation is 1.0134
Our exact value is 1.0000
With an error of -0.0134.

t = 0.25
alpha = 0.001159
Our approximation is 1.3033
Our exact value is 1.2840
With an error of -0.0193.

t = 0.50
```

```
alpha = 0.001159
Our approximation is 1.6625
Our exact value is 1.6487
With an error of -0.0138.

t = 0.75
alpha = 0.001159
Our approximation is 2.1076
Our exact value is 2.1170
With an error of 0.0094.

t = 1.00
alpha = 0.001159
Our approximation is 2.6593
Our exact value is 2.7183
With an error of 0.0589.

Would you like to run again?
Please type 'y' or 'n'.
 n
```